L^2 TORSION WITHOUT THE DETERMINANT CLASS CONDITION AND EXTENDED L^2 COHOMOLOGY

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ABSTRACT. We associate determinant lines to objects of the extended abelian category built out of a von Neumann category with a trace. Using this we suggest constructions of the combinatorial and the analytic L^2 torsions which, unlike the work of the previous authors, requires no additional assumptions; in particular we do not impose the determinant class condition. The resulting torsions are elements of the determinant line of the extended L^2 cohomology. Under the determinant class assumption the L^2 torsions of this paper specialize to the invariants studied in our previous work [6]. Applying a recent theorem of D. Burghelea, L. Friedlander and T. Kappeler [3] we obtain a Cheeger - Müller type theorem stating the equality between the combinatorial and the analytic L^2 torsions.

1. INTRODUCTION

M. Atiyah [1] was the first who used the concept of von Neumann dimension in algebraic topology. He applied this notion to measure the size of the space of harmonic forms on the universal cover. Later S. Novikov and M. Shubin [30] suggested more sophisticated spectral invariants which are now known as the Novikov – Shubin invariants. These invariants measure the size of the space of the forms on the universal cover which are only "nearly" harmonic. Gromov and Shubin [18] proved the homotopy invariance of the Novikov – Shubin invariants.

Around 1986 Novikov and Shubin proposed to study L^2 torsion as a von Neumann analogue of the classical Reidemeister torsion. A detailed treatment of the theory of L^2-torsion was first done in 1992 by A. Carey, V. Mathai and J. Lott, see [7], [25], [20]. In these papers the L^2 torsion is a positive real number which is defined under the assumption that, firstly, the von Neumann Betti numbers vanish and, secondly, certain *"determinant class condition"* is satisfied. The latter condition can be expressed in term of the spectral density function of the Laplacian; it is satisfied if the Novikov – Shubin invariants are positive.

Paper [6] showed how one may modify the construction of the L^2 torsion so that the first of the above mentioned assumptions (the vanishing of the reduced L^2 cohomology) becomes superfluous. Namely, it was shown in [6] that one may naturally associate determinant lines to finitely generated Hilbertian modules over von Neumann algebras. Nonzero elements of these determinant lines are volume forms on Hilbertian modules. The L^2 torsion is then

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defined in [6] as an element of the determinant line of the reduced L^2 cohomology. However, the determinant class assumption was still present in [6].

Let us mention here a standing conjecture (raised by V. Mathai and J. Lott) that the Novikov – Shubin invariants are always positive for the special case of the regular representation $\pi \to \ell^2(\pi)$ of the fundamental group π of a compact polyhedron. At present this conjecture remains open. However, in the more general setting of Hilbertian representations which we study in this paper, the determinant class condition may be easily violated (even when the underlying manifold is the circle). Therefore removing the determinant class assumption seems to be important for the theory of L^2 torsion.

The main goal of this paper is to define the L^2 torsion without requiring the determinant class condition. It turns out that the L^2 torsion (both combinatorial and analytic) is well defined as an element of the determinant line associated to the extended L^2 cohomology. We also prove here the Cheeger – Müller type theorem stating coincidence between the combinatorial and the analytic torsions.

The notion of the extended L^2 cohomology was first introduced by M. Farber in [11], [12]. The main idea was to embed the category of Hilbertian representation into an abelian category and to view the cohomology of the chain complex of L^2 chains of Atiyah [1] as an object of the extended abelian category. The work [11], [12] was inspired by the general categorical study of P. Freyd [16]. The approach of [11], [12] compared with the notion of the reduced cohomology gives a bigger homological object: the extended cohomology contains the reduced cohomology and carries the information about the von Neumann Betti numbers; it also determines the Novikov – Shubin invariants and some other new invariants, see [14]. The great advantage of the extended cohomology is that it allows to use the well-developed formalism of abelian categories to compute the L^2 cohomology. An algebraic analogue of the extended cohomology theory of [11], [12] was later suggested by W. Lück [22].

The present paper gives an additional reason to believe that the extended cohomology, allowing to unify the theory of L^2 torsion, is the right object to study. We show how to associate a determinant line $\det \mathcal{X}$ with any object $\mathcal{X} = (\alpha : A' \to A)$ of the extended category. If α is injective, an element of $\det \mathcal{X}$ can be represented as a ratio

$$\frac{\langle \, , \, \rangle}{\langle \, , \, \rangle_1} \in \det \mathcal{X}$$

where $\langle \, , \, \rangle$ and $\langle \, , \, \rangle_1$ are admissible scalar products on A and A' respectively. The determinant line $\det \mathcal{X}$ is the tensor product of the determinant lines associated to the projective and torsion parts of $\mathcal{X}$. We show in this paper that determinant line of some torsion objects (they are called τ-trivial) can be canonically trivialized. Moreover, the determinant class condition means precisely that the torsion part of the extended cohomology is τ-trivial. This fully explains the role of the determinant class assumption.

Our proof of the Cheeger – Müller type theorem stating the equality between the analytic and the combinatorial torsions is based on a recent theorem of D. Burghelea, L. Friedlander and T. Kappeler [3] which uses the language of the relative torsion.

Since this paper is already quite long we decided to publish the discussion of examples and the applications elsewhere.

Contents

2. Hilbertian von Neumann categories and Fuglede-Kadison determinants

This section contains preliminary material which will be used later in this paper.

First we recall the notions of a von Neumann category and of a trace on a von Neumann category; here we follow papers [13, 15]. To understand the main constructions and results of the paper the reader may keep in mind the simplest and the most important for applications example of the category of Hilbertian modules over a von Neumann algebra, cf. Subsection 2.5.2. We recall also (following [6]) how one uses a trace to define the Fuglede-Kadison determinant.

2.1. **Hilbertian spaces.** Recall that *a Hilbertian space* (cf. [31]) is a topological vector space H, which is isomorphic to a Hilbert space in the category of topological vector spaces. In other words, there exists a scalar product on H, such that H with this scalar product is a Hilbert space with the originally given topology.

Scalar products with the above property are called *admissible*. Given one admissible scalar product $\langle \, , \, \rangle$ on H, any other admissible scalar product is given by

$$\langle x, y \rangle_1 \; = \; \langle Ax, y \rangle, \quad x, y \in H,$$

where $A : H \to H$ is an invertible positive operator $A^* = A, \quad A > 0.$

Hilbertian spaces naturally appear as Sobolev spaces of sections of vector bundles, cf. [31].

Let us denote by *Hilb* the category of Hilbertian spaces and continuous linear maps.

If H is a Hilbertian space, H^* will denote its conjugate, i.e. the space of all anti-linear continuous functionals on H (bounded $\mathbb{R}$-linear maps $\phi : H \to \mathbb{C}$, such that $\phi(\lambda h) = \overline{\lambda}\phi(h)$ for all $\lambda \in \mathbb{C}$ and $h \in H$; here the bar denotes the complex conjugation). We consider the action of $\mathbb{C}$ on H^* given by $(\lambda \cdot \phi)(h) = \phi(\overline{\lambda} \cdot h)$ for all $h \in H$. The canonical isomorphism

$$H \;\to\; (H^*)^* \;=\; H^{**} \tag{2.1}$$

is given by $h \mapsto (\phi \mapsto \overline{\phi(h)})$, where $h \in H$, and $\phi \in H^*$.

The following definition was suggested in [15].

Definition 2.2. *A Hilbertian von Neumann category is an additive subcategory $\mathcal{C}$ of Hilb with the following properties:*

(1) *for any $H \in \mathrm{Ob}(\mathcal{C})$ the dual space H^* is also an object of $\mathcal{C}$ and there is a $\mathcal{C}$-isomorphism $\phi : H \to H^*$ such that the formula*

$$\langle x, y \rangle \;=\; \phi(x)(y), \qquad x, y \in H \tag{2.2}$$

defines an admissible scalar product on H;

(2) *for any $H \in \mathrm{Ob}(\mathcal{C})$ the isomorphism (2.1) also belongs to $\mathcal{C}$;*

(3) *the adjoint of any morphism in $\mathcal{C}$ also belongs to $\mathcal{C}$;*

(4) *the kernel $\ker f = \{x \in H; f(x) = 0\}$ of any morphism $f : H \to H'$ in $\mathcal{C}$ and the natural inclusion $\ker f \to H$ belong to $\mathcal{C}$;*

(5) *for any $H, H' \in \mathrm{Ob}(\mathcal{C})$ the set of morphisms*

$$\mathrm{Hom}_{\mathcal{C}}(H, H') \;\subset\; \mathrm{Hom}_{Hilb}(H, H') \;=\; \mathcal{L}(H, H')$$

is a linear subspace closed with respect to the weak topology.

Thus, objects of $\mathcal{C}$ have structure of Hilbertian spaces and possibly some additional structure; morphisms of $\mathcal{C}$ are (faithfully) represented by bounded linear maps.

Condition (5) is similar to the well-known condition in the definition of von Neumann algebras, which explains our term.

Definition 2.3. *Let H be an object of a Hilbertian von Neumann category $\mathcal{C}$. An admissible scalar product on H is called $\mathcal{C}$-admissible if it is given by the equation (2.2) for some $\phi \in \mathrm{Hom}_{\mathcal{C}}(H, H^*)$.*

If $\langle \ , \ \rangle$ and $\langle \ , \ \rangle_1$ are two $\mathcal{C}$-admissible scalar products then there exists an invertible positive operator $A \in \mathrm{Hom}_{\mathcal{C}}(H, H)$ such that

$$\langle v, w \rangle_1 \;=\; \langle Av, w \rangle, \qquad \text{for all} \quad v, w \in H. \tag{2.3}$$

Note also, that given any object $H \in \mathrm{Ob}(\mathcal{C})$, a choice of a $\mathcal{C}$-admissible scalar product on H determines an involution on the algebra $\mathrm{Hom}_{\mathcal{C}}(H, H)$ (adjoint operator) and the space $\mathrm{Hom}_{\mathcal{C}}(H, H)$ considered with this involution is a von Neumann algebra.

The conditions (1) - (5) of Definition 2.2 imply the following properties:

(6) *The closure of the image* $\mathrm{cl}(\mathrm{im}\, f)$ *of any morphism* $f : H \to H'$ *in* $\mathcal{C}$ *and also the natural projection* $H' \to H'/\mathrm{cl}(\mathrm{im}\, f)$ *belong to* $\mathcal{C}$.

(7) *Suppose that* $H' \subset H$ *is a closed subspace. If* H', H *and the inclusion* $H' \to H$ *belong to* $\mathcal{C}$ *then the orthogonal complement* $H'^{\perp}$ *with respect to a* $\mathcal{C}$*-admissible scalar product on* H *and the inclusion* $H'^{\perp} \to H$ *belong to* $\mathcal{C}$.

2.4. **Finite von Neumann categories.** The following definitions were suggested in [13]. An object H of a Hilbertian von Neumann category $\mathcal{C}$ is called *finite* if every injective morphism $f \in \mathrm{Hom}_{\mathcal{C}}(H, H)$ has a dense image. A Hilbertian von Neumann category is called *finite* iff all its objects are finite.

2.5. **Examples of von Neumann categories.**

2.5.1. *Hilbertian representations.* The simplest and the most important example of a Hilbertian von Neumann category is the following (see [15], §2.3). Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ with involution, which on $\mathbb{C}$ coincides with the complex conjugation. *A Hilbertian representation of* $\mathcal{A}$ is a Hilbertian topological vector space H supplied with a left action $\mathcal{A} \to \mathcal{L}(H, H)$ of $\mathcal{A}$ by continuous linear maps. A morphism $f : H \to H'$ between two Hilbertian representations of $\mathcal{A}$ is defined as a bounded linear map commuting with the action of the algebra $\mathcal{A}$. We denote the obtained category by $\mathcal{C}_{\mathcal{A}}$.

There is a canonical duality[1] in the category of all Hilbertian representations of a given $*$-algebra $\mathcal{A}$. Namely, given a Hilbertian representation H, consider the space H^* of all anti-linear continuous functionals on H. Consider the following action of $\mathcal{A}$ on H^*: if $\phi \in H^*$ and $\lambda \in \mathcal{A}$ then $(\lambda \cdot \phi)(h) = \phi(\lambda^* \cdot h)$ for all $h \in H$. Here λ^* denotes the involution of $\lambda \in \mathcal{A}$. The canonical isomorphism

$$H \;\to\; H^{**}$$

is given by $h \mapsto (\phi \mapsto \overline{\phi(h)})$, where $h \in H$, and $\phi \in H^*$. Category of all Hilbertian representations of $\mathcal{A}$ is a von Neumann category.

2.5.2. *Hilbertian modules.* Let now $\mathcal{A}$ be a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau : \mathcal{A} \to \mathbb{C}$. In this case there is an important subcategory of the category $\mathcal{C}_{\mathcal{A}}$ (described in the previous section).

Let $*$ be the involution in $\mathcal{A}$. By $\ell^2(\mathcal{A}) = \ell^2_{\tau}(\mathcal{A})$ we denote the completion of $\mathcal{A}$ with respect to the scalar product $\langle a, b \rangle = \tau(b^* a)$, for $a, b \in \mathcal{A}$, determined by the trace τ. A *(projective) Hilbertian* $\mathcal{A}$*-module* M is a Hilbertian representation of $\mathcal{A}$ such that there exists a continuous $\mathcal{A}$-linear embedding of M into $\ell^2(\mathcal{A}) \otimes H$, for some Hilbert space H. Note that this embedding is not part of the structure. A Hilbertian module M is said to be *finitely generated* if it admits an embedding $M \to \ell^2(\mathcal{A}) \otimes H$ as above with finite dimensional H.

We denote by $\mathcal{H}_{\mathcal{A}}$ the full subcategory of $\mathcal{C}_{\mathcal{A}}$, whose objects are Hilbertian $\mathcal{A}$-modules. We denote by $\mathcal{H}^f_{\mathcal{A}}$ the full subcategory of finitely generated modules in $\mathcal{H}_{\mathcal{A}}$.

Note that the algebra $\mathrm{Hom}_{\mathcal{H}_{\mathcal{A}}}(M, M)$ coincides with the commutant $\mathcal{B} = \mathcal{B}_{\mathcal{A}}(M)$ of M.

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[1] A notion of duality in a von Neumann category was studied in detail in [15].

$\mathcal{H}_{\mathcal{A}}$ is a von Neumann category; $\mathcal{H}_{\mathcal{A}}^{f}$ is a finite von Neumann category.

2.5.3. *Families of Hilbert spaces.* Let Z be a locally compact Hausdorff space and let μ be a positive Radon measure on Z. Let $\mathcal{A}$ denote the algebra $L_{\mathbb{C}}^{\infty}(Z, \mu)$ (the space of essentially bounded μ-measurable complex valued functions on Z, in which two functions equal locally almost everywhere, are identical). The involution on $\mathcal{A}$ is given by the complex conjugation. We will construct a category $\mathcal{C}(Z, \mu)$ of Hilbert representations of $\mathcal{A}$ as follows. The objects of $\mathcal{C}(Z, \mu)$ are in one-to-one correspondence with the μ-measurable fields of finite-dimensional Hilbert spaces $\xi \to \mathcal{H}(\xi)$ over (Z, μ), cf. [10], part II, chapter 1. For any such measurable field of Hilbert spaces, the corresponding Hilbert space is the direct integral

$$H = \int^{\oplus} \mathcal{H}(\xi) \, d\mu(\xi) \tag{2.4}$$

defined as in [10], part II, chapter 1. The algebra $\mathcal{A}$ acts on the Hilbert space (2.4) by pointwise multiplication.

Suppose that we have two μ-measurable finite-dimensional fields of Hilbert spaces $\xi \to \mathcal{H}(\xi)$ and $\xi \to \mathcal{H}'(\xi)$ over Z. Then we have two corresponding Hilbert spaces, H and H', given as direct integrals (2.4). We define the set of morphisms $\operatorname{Hom}_{\mathcal{C}(Z,\mu)}(H, H')$ as the set of all bounded linear maps $H \to H'$ given by *decomposable linear maps*

$$T = \int^{\oplus} T(\xi) \, d\mu(\xi), \tag{2.5}$$

where $T(\xi)$ is *an essentially bounded measurable field of linear maps* $T(\xi) : \mathcal{H}(\xi) \to \mathcal{H}'(\xi)$, cf. [10], part II, chapter 2.

It is shown in [13, §2.6] that the above construction defines a finite von Neumann category.

One can also consider a full subcategory $\mathcal{C}^{f}(Z, \mu)$ of $\mathcal{C}(Z, \mu)$, which consists of measurable fields of finite dimensional Hilbert spaces $\mathcal{H}(\xi)$ over Z such that the dimensions of $\mathcal{H}(\xi)$ are essentially bounded.

Other examples of von Neumann categories can be found in [13, §2].

2.6. **Traces on von Neumann categories.** Let $\mathcal{C}$ be a Hilbertian von Neumann category. The following definition was suggested in §5.8 of [13]:

Definition 2.7. *A trace on $\mathcal{C}$ is a function, denoted τ, which assigns to each object $\mathcal{H} \in \operatorname{Ob}(\mathcal{C})$ a linear functional*

$$\tau_{\mathcal{H}} : \operatorname{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H}) \to \mathbb{C}$$

on the von Neumann algebra $\operatorname{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H})$, such that for any pair of objects $\mathcal{H}_1, \mathcal{H}_2 \in \operatorname{Ob}(\mathcal{C})$ the corresponding traces $\tau_{\mathcal{H}_1}$ and $\tau_{\mathcal{H}_2}$ are compatible in the following sense: if $f \in \operatorname{hom}_{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_2)$ and $g \in \operatorname{hom}_{\mathcal{C}}(\mathcal{H}_2, \mathcal{H}_1)$ then

$$\tau_{\mathcal{H}_1}(gf) = \tau_{\mathcal{H}_2}(fg).$$

The trace τ is called non-negative if $\tau_{\mathcal{H}}(e)$ is real and non-negative for any idempotent $e \in \operatorname{hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H})$, $e^2 = e$. One says that that a trace τ on a von Neumann category is

normal (or faithful) iff for each non-zero $\mathcal{H} \in \mathrm{Ob}(\mathcal{C})$ the trace $\tau_{\mathcal{H}}$ on the von Neumann algebra $\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{H})$ is normal (faithful).

If $f \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$ is given by a 2×2 matrix (f_{ij}), where $f_{ij} : \mathcal{H}_i \to \mathcal{H}_j$, $i, j = 1, 2$, then

$$\tau_{\mathcal{H}_1 \oplus \mathcal{H}_2}(f) = \tau_{\mathcal{H}_1}(f_{11}) + \tau_{\mathcal{H}_2}(f_{22}) \tag{2.6}$$

as follows easily from the above definition.

To simplify the notation we will often write τ for τ_H.

2.8. Example: Trace on the category of finitely generated Hilbertian modules.
Let $\mathcal{A}$ be a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau : \mathcal{A} \to \mathbb{C}$. Consider the category $\mathcal{H}_{\mathcal{A}}^f$ of finitely generated $\mathcal{A}$-Hilbertian modules, cf. Section 2.5.2. There is a canonical trace on this category, which we will also denote by τ, cf. [6, Proposition 1.8]. Let us briefly recall the construction of this trace.

Suppose first that M is *free*, that is, M is isomorphic to $\ell^2(\mathcal{A}) \otimes \mathbb{C}^k$ for some k. Then $\mathrm{Hom}_{\mathcal{H}_{\mathcal{A}}^f}(M, M)$ can be identified with the algebra of $k \times k$-matrices with entries in $\mathcal{A}$, acting from the right on $\ell^2(\mathcal{A}) \otimes \mathbb{C}^k$ (the last module is viewed as the set of row-vectors with entries in $\ell^2(\mathcal{A})$). If $\alpha \in \mathcal{B}$ is represented by a $k \times k$ matrix (α_{ij}), then one define

$$\tau_M(\alpha) = \sum_{i=1}^{k} \tau(\alpha_{ii}).$$

This gives a trace on $\mathrm{Hom}_{\mathcal{H}_{\mathcal{A}}^f}(M, M)$ which satisfies all necessary conditions.

If M is not free, then we can embed it in a free module as a closed $\mathcal{A}$-invariant subspace. Then $\mathrm{Hom}_{\mathcal{H}_{\mathcal{A}}^f}(M, M)$ can be identified with a left ideal in the $k \times k$-matrix algebra with entries in $\mathcal{A}$ and the trace described in the previous paragraph restricts to this ideal and determines a trace on $\mathrm{Hom}_{\mathcal{H}_{\mathcal{A}}^f}(M, M)$. One then shows that the obtained trace does not depend on the embedding of M in a free module.

2.9. Example: Trace on the category of families of Hilbert spaces.
Let Z be a locally compact Hausdorff space endowed with a positive Radon measure μ. Consider the von Neumann category $\mathcal{C}^f(Z, \mu)$ of measurable fields of finite dimensional Hilbert spaces $\mathcal{H}(\xi)$ over Z such that the dimensions of $\mathcal{H}(\xi)$ are essentially bounded, cf. Subsection 2.5.3. This category has traces. Let ν be a positive measure on Z which is absolutely continuous with respect to μ and such that $\nu(Z) < \infty$. Then ν determines the following trace on this von Neumann category

$$\tau_{\nu}(T) = \int_Z \mathrm{Tr}(T(\xi)) \, d\nu.$$

where T is a morphism given by formula (2.5) and Tr denotes the usual finite dimensional trace.

2.10. **The Fuglede-Kadison determinant.** Suppose $\mathcal{C}$ is a Hilbertian von Neumann category endowed with a non-negative, normal, and faithful trace. Let M be an object in $\mathcal{C}$. Denote by $\mathrm{GL}(M)$ the group of all invertible elements of $\mathrm{Hom}_{\mathcal{C}}(M, M)$. We will consider the norm topology on $\mathrm{GL}(M)$; with this topology it is a Banach Lie group. Its Lie algebra can be identified with $\mathrm{Hom}_{\mathcal{C}}(M, M)$. The trace τ_M is a homomorphism of the Lie algebra $\mathrm{Hom}_{\mathcal{C}}(M, M)$ into the abelian Lie algebra $\mathbb{C}$. By the standard theorems, it defines a group homomorphism of the universal covering group of $\mathrm{GL}(M)$ into $\mathbb{C}$. This approach leads to following construction of the Fuglede-Kadison determinants, cf. [9].

Given an invertible operator $A \in \mathrm{GL}(M)$, find a continuous piecewise smooth path $A_t \in \mathrm{GL}(M)$ with $t \in [0, 1]$, such that $A_0 = I$ and $A_1 = A$ (it is well known that the group $\mathrm{GL}(M)$ is pathwise connected, cf. [10]). Then define

$$\log \mathrm{Det}_\tau(A) = \int_0^1 \Re\tau\left[A_t^{-1} A_t' \right] dt, \tag{2.7}$$

where $\Re$ denotes the real part.

It has been shown in [6] that the integral does not depend on the choice of the path, joining A with the identity I, and that one has the following [2] theorem:

Theorem 2.11. *The function above,*

$$\mathrm{Det}_\tau : \mathrm{GL}(M) \longrightarrow \mathbb{R}^{>0} \tag{2.8}$$

called the Fuglede–Kadison determinant, satisfies:

(a) Det_τ *is a group homomorphism, that is,*

$$\mathrm{Det}_\tau(AB) = \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B) \quad for \quad A, B \in \mathrm{GL}(M); \tag{2.9}$$

(b)

$$\mathrm{Det}_\tau(\lambda I) = |\lambda|^{\dim_{\mathcal{A}}(M)} \quad for \quad \lambda \in \mathbb{C}, \lambda \neq 0; \tag{2.10}$$

here $I \in \mathrm{GL}(M)$ denotes the identity operator;

(c)

$$\mathrm{Det}_{\lambda\tau}(A) = \mathrm{Det}_\tau(A)^\lambda \quad for \quad \lambda \in \mathbb{R}^{>0}; \tag{2.11}$$

(d) Det_τ *is continuous as a map $\mathrm{GL}(M) \to \mathbb{R}^{>0}$, where $\mathrm{GL}(M)$ is supplied with the norm topology;*

(e) *If A_t for $t \in [0, 1]$ is a continuous piecewise smooth path in $\mathrm{GL}(M)$ then*

$$\log\left[\frac{\mathrm{Det}_\tau(A_1)}{\mathrm{Det}_\tau(A_0)} \right] = \int_0^1 \Re\mathrm{Tr}[A_t^{-1} A_t'] dt. \tag{2.12}$$

Here A_t' denotes the derivative of A_t with respect to t.

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[2]In [6] the theorem is stated only for the category of finitely generated Hilbertian representation of a von Neumann algebra, cf. Subsection 2.5.1. However the proof works for arbitrary von Hilbertian Neumann category with trace without any changes.

(f) *Let M and N be two objects of $\mathcal{C}$, and $A \in \mathrm{GL}(M)$ and $B \in \mathrm{GL}(N)$ two invertible automorphisms, and let $\gamma : N \to M$ be a homomorphism. Then the map given by the matrix*

$$\begin{pmatrix} A & \gamma \\ 0 & B \end{pmatrix}$$

belongs to $\mathrm{GL}(M \oplus N)$ and

$$\mathrm{Det}_\tau \begin{pmatrix} A & \gamma \\ 0 & B \end{pmatrix} \;=\; \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B). \tag{2.13}$$

As an example consider an object $M \in \mathrm{Ob}(\mathcal{C})$ and a $\mathcal{C}$-automorphism $A : M \to M$. Fix an admissible scalar product on M. Assume that A is positive and self-adjoint with respect to this scalar product. Let $A = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of A. Then

$$\mathrm{Det}_\tau(A) = \exp \left[\int_0^\infty \ln(\lambda) d\phi(\lambda) \right] \tag{2.14}$$

where $\phi(\lambda) = \tau(E_\lambda)$ is the spectral density function of A.

3. Determinant line of a Hilbertian module

In this section we present a construction of the determinant line associated to an object in a Hilbertian von Neumann category with trace, which essentially repeats a construction given in [6]. It will be used later in Section 5 in the constructions of the determinant line assocoated to an object in the extended category, and then in Sections 10 and 12 to define the combinatorial and the analytic L^2-torsions invariants associated to polyhedra and compact manifolds respectively.

3.1. **The definition.** Let $\mathcal{C}$ be a Hilbertian von Neumann category endowed with a non-negative, normal, and faithful trace, cf. Definitions 2.2 and 2.7 [3] . One associates (in a canonical way) with any object $M \in \mathrm{Ob}\,(\mathcal{C})$ an oriented real line which is denoted $\det M$; it is the *determinant line of M*. This construction generalizes the determinant line of a finite dimensional vector space.

Define $\det M$ as a real vector space generated by symbols $\langle \, , \, \rangle$, one for any $\mathcal{C}$-admissible scalar product on M, subject to the following relations: for any pair $\langle \, , \, \rangle_1$ and $\langle \, , \, \rangle_2$ of $\mathcal{C}$-admissible scalar products on M one writes the following relation

$$\langle \, , \, \rangle_2 \;=\; \mathrm{Det}_\tau(A)^{-1/2} \cdot \langle \, , \, \rangle_1 \tag{3.1}$$

where $A \in \mathrm{GL}(M)$ is such that

$$\langle v, w \rangle_2 \;=\; \langle Av, w \rangle_1$$

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[3]We emphasize that for most of our applications it is enough to consider the special case of the category $\mathcal{C} = \mathcal{H}_\mathcal{A}^f$ of finitely generated Hilbertian modules, cf. Subsection 2.5.2.

for all $v, w \in M$. Here the transition operator $A \in \mathrm{Hom}_{\mathcal{C}}(M, M)$ is invertible, cf. (2.3) and $\mathrm{Det}_{\tau}(A)$ denotes the Fuglede-Kadison determinant of A constructed in Subsection 2.10 with the aid of the trace τ on the von Neumann category $\mathcal{C}$.

Then, cf. [6, Section 2.1], det M *is the one-dimensional real vector space generated by the symbol* $\langle\ ,\ \rangle$ *of any* $\mathcal{C}$-*admissible scalar product on* M.

Note also, that the real line $\det M$ has *the canonical orientation*, since the transition coefficients $\mathrm{Det}_{\tau}(A)^{-1/2}$ are always positive. Thus we may speak of *positive and negative* elements of $\det M$. The set of all positive elements of $\det M$ will be denoted $\det_{+}(M)$.

We will think of elements of $\det M$ as "densities" on M. If M is a trivial object, $M = 0$, then we set $\det M = \mathbb{R}$, by definition.

3.2. **Determinant line of a direct sum.** Given two objects M and N in $\mathcal{C}$, and a pair $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ of $\mathcal{C}$-admissible scalar products on M and N correspondingly, we may obviously define the scalar product $\langle\ ,\ \rangle_M \oplus \langle\ ,\ \rangle_N$ on the direct sum $M \oplus N$. This defines an isomorphism

$$\phi : \det M \otimes \det N \ \xrightarrow{\ \sim\ }\ \det(M \oplus N). \tag{3.2}$$

Using Theorem 2.11 (properties (a) and (f)), it is easy to show that *this homomorphism is canonical*, that is, it does not depend on the choice of the metrics $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$. From the description given above it is clear that the homomorphism (3.2) preserves the orientations.

3.3. **Push-forward of determinant lines.** Note that, *any isomorphism* $f : M \to N$ *between objects* M, N *of* $\mathcal{C}$ *induces canonically an isomorphism of the determinant lines*

$$f_* : \det M \to \det N. \tag{3.3}$$

Moreover, the induced map f_* *preserves the orientations of the determinant lines.*

Indeed, if $\langle\ ,\ \rangle_M$ is an $\mathcal{C}$-admissible scalar product on M, then we set

$$f_*(\langle\ ,\ \rangle_M) \ =\ \langle\ ,\ \rangle_N, \tag{3.4}$$

where $\langle\ ,\ \rangle_N$ is the scalar product on N given by

$$\langle v, w\rangle_N \ =\ \langle f^{-1}(v), f^{-1}(w)\rangle_M \qquad \text{for}\quad v, w \in N$$

(this scalar product is $\mathcal{C}$-admissible since $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ is an isomorphism). This definition does not depend on the choice of the scalar product $\langle\ ,\ \rangle_M$ on M: if we have a different $\mathcal{C}$-admissible scalar product $\langle\ ,\ \rangle'_M$ on M, where $\langle v, w\rangle'_M = \langle Av, w\rangle_M$ with $A \in \mathrm{GL}(M)$, then the induced scalar product on N will be $\langle v, w\rangle'_N = \langle (f^{-1}Af)v, w\rangle_N$ and our statement follows from property (a) of the Fuglede-Kadison determinant, cf. Theorem 2.11.

3.4. **Calculation of the push-forward.** Suppose $f : M \to N$ is an isomorphism and fix $\mathcal{C}$-admissible scalar products $\langle\ ,\ \rangle_M$, $\langle\ ,\ \rangle_N$ on M and N respectively (note that we don't assume any more that the equality (3.4) is satisfied). Let $f^* : N \to M$ be the adjoint of f with respect to these scalar products. Then, for every $v, w \in N$ we have

$$\langle\, f^{-1}(v), f^{-1}(w)\,\rangle_M \ =\ \langle\, (ff^*)^{-1}(v), w\,\rangle_N.$$

Comparing this equation with (3.4) and (3.1), we obtain

$$f_*(\langle\ ,\ \rangle_M) = \sqrt{\mathrm{Det}_\tau(ff^*)}\cdot\langle\ ,\ \rangle_N \qquad (3.5)$$

In particular, in the case $M = N$ we obtain the following

Proposition 3.5. *If $f : M \to M$ is an automorphism of $M \in \mathrm{Ob}\,(\mathcal{C})$, then the induced homomorphism $f_* : \det M \to \det M$ coincides with the multiplication by $\mathrm{Det}_\tau(f) \in \mathbb{R}^{>0}$.*

3.6. **Composition of the push-forwards.** It is obvious from the definition, that the construction of the push-forward is *functorial*: if $f : M \to N$ and $g : N \to L$ are two isomorphisms between objects of $\mathcal{C}$ then

$$(g \circ f)_* = g_* \circ f_*. \qquad (3.6)$$

Proposition 3.7. *Any exact sequence*

$$0 \to M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \to 0 \qquad (3.7)$$

of objects in $\mathcal{C}$ determines canonically an isomorphism

$$\psi_{\alpha,\beta} : \det M \ \to \ \det M' \otimes \det M'' \qquad (3.8)$$

preserving the orientations of the lines.

Proof. The map $\psi_{\alpha,\beta}$ is constructed as follows. Any admissible scalar product $\langle\cdot,\cdot\rangle_M$ on M defines admissible scalar products $\langle\cdot,\cdot\rangle_{M'}$ on M' and $\langle\cdot,\cdot\rangle_{M''}$ on M'' as follows:

$$\langle x', y'\rangle_{M'} = \langle\alpha(x'), \alpha(y')\rangle_M, \quad \langle x'', y''\rangle_{M''} = \langle\gamma(x''), \gamma(y'')\rangle_M$$

for $x', y' \in M'$ and $x'', y'' \in M''$. Here $\gamma : M'' \to M$ is such that $\beta \circ \gamma = 1_{M''}$ and $\langle\alpha(x'), \gamma(x'')\rangle_M = 0$ for any $x' \in M'$ and $x'' \in M''$. These conditions determine γ uniquely; in fact γ identifies M'' with the orthogonal complement $\mathrm{im}(\alpha)^\perp$ with respect to $\langle\cdot,\cdot\rangle_M$.

Next we show that $\psi_{\alpha,\beta}$ is well-defined, i.e. the class of $\langle\ ,\ \rangle_{M'} \oplus \langle\ ,\ \rangle_{M''}$ in the determinant line $\det(M' \oplus M'')$ depends only on the class of the scalar product $\langle\ ,\ \rangle_M$ in $\det(M)$. Consider an automorphism $h : M \to M$ and the induced scalar product $\langle x, y\rangle'_M = \langle h(x), h(y)\rangle_M$ on M. Let $\gamma : M'' \to M$ be the splitting as above. The splitting $\bar\gamma : M'' \to M$ corresponding to the the new scalar product $\langle\cdot,\cdot\rangle'_M$ equals $\bar\gamma = \gamma + \alpha \circ f$, where $f : M'' \to M'$ is a morphism which is uniquely determined by the data. One has

$$[\alpha, \bar\gamma] = [\alpha, \gamma] \circ \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}. \qquad (3.9)$$

Here $[\alpha, \gamma]$ denotes a morphism $M' \oplus M'' \to M$ whose restrictions to M' and M'' are α and γ, correspondingly. The symbol $[\alpha, \bar\gamma]$ has a similar meaning. The matrix $\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}$ represents a morphism $M' \oplus M'' \to M' \oplus M''$. Let $\xi, \xi' \in \det(M' \oplus M'')$ be such that

$$[\alpha, \gamma]_*(\xi) = \langle\ ,\ \rangle_M, \quad [\alpha, \bar\gamma](\xi') = \langle\ ,\ \rangle'_M.$$

Using the properties of the push-forwards applied to (3.9) and Theorem 2.11, we find

$$[\alpha, \bar{\gamma}](\xi) = \langle \ , \ \rangle_M = \mathrm{Det}_\tau(h)^{1/2} \langle \ , \ \rangle'_M.$$

Thus, $\xi' = \mathrm{Det}_\tau(h)^{-1/2} \cdot \xi \in \det(M' \oplus M'')$. In the determinant line $\det(M)$ one has the relation $\langle \ , \ \rangle'_M = \mathrm{Det}_\tau(h)^{-1/2} \cdot \langle \ , \ \rangle_M$; hence it follows that our definitions $\xi = \psi_{\alpha,\beta}(\langle \ , \ \rangle_M)$ and $\xi' = \psi_{\alpha,\beta}(\langle \ , \ \rangle'_M)$ are compatible.   □

   The following lemma will be used in Section 5.

Lemma 3.8. *Under the conditions of Proposition 3.7 let $h : M \to M$ be an automorphism of M. Then the automorphism $\psi_{h \circ \alpha, \beta \circ h^{-1}}$ corresponding to the exact sequence*

$$0 \to M' \xrightarrow{h \circ \alpha} M \xrightarrow{\beta \circ h^{-1}} M'' \to 0 \tag{3.10}$$

coincides with $\mathrm{Det}_\tau(h)^{-1} \cdot \psi_{\alpha,\beta}$.

Proof. Fix an admissible scalar product $\langle \cdot, \cdot \rangle_M$ on M. Let $\gamma : M'' \to M$ be such that $\beta \circ \gamma = 1_{M''}$ and $\langle \alpha(x'), \gamma(x'') \rangle_M = 0$ for any $x' \in M'$ and $x'' \in M''$. Then

$$\psi_{\alpha,\beta}(\langle \ , \ \rangle_M) = \xi \in \det(M' \oplus M'')$$

where ξ satisfies

$$[\alpha, \gamma]_*(\xi) = \langle \ , \ \rangle_M. \tag{3.11}$$

This is the definition of $\psi_{\alpha,\beta}$, see above. Now consider the sequence (3.10). Fix the following scalar product on M:

$$\langle x, y \rangle'_M = \langle h^{-1}(x), h^{-1}(y) \rangle_M, \quad x, y, M.$$

The corresponding splitting is $h \circ \gamma : M'' \to M$. Hence

$$\psi_{h \circ \alpha, \beta \circ h^{-1}}(\langle \ , \ \rangle'_M) = \xi', \tag{3.12}$$

where

$$[h \circ \alpha, h \circ \gamma]_*(\xi') = \langle \ , \ \rangle'_M. \tag{3.13}$$

In the determinant line $\det(M)$ we have

$$\langle \ , \ \rangle'_M = \mathrm{Det}_\tau(h) \cdot \langle \ , \ \rangle_M. \tag{3.14}$$

Using the properties of the push-forwards we obtain

$$[h \circ \alpha, h \circ \gamma]_*(\xi') = h_* \circ [\alpha, \gamma]_*(\xi') = \mathrm{Det}_\tau(h) \cdot [\alpha, \gamma]_*(\xi'). \tag{3.15}$$

Combining (3.11), (3.12), (3.13), (3.14), (3.15) one obtains $\xi = \xi'$. This proves that $\psi_{h \circ \alpha, \beta \circ h^{-1}} = \mathrm{Det}_\tau(h)^{-1} \cdot \psi_{\alpha,\beta}$ as stated.

     □

4. Abelian extension of a von Neumann category

M. Farber in [12] and [13] constructs an abelian category called the *extended category of Hilbertian modules*. This abelian category contains a given von Neumann category $\mathcal{C}$ as a full subcategory of its projective objects. The construction of the extended category of [12], [13] was inspired by the earlier work of P. Freyd [16] on embedding of additive categories into abelian categories.

Any object of the extended category splits naturally into projective and torsion components; the projective part has the von Neumann dimension as an invariant while the Novikov-Shubin invariant [30] depends only on the torsion part.

A brief description of Farber's construction is given below. We refer to the paper [13] for more details.

Definition 4.1. *Let $\mathcal{C}$ be a Hilbertian von Neumann category. The* extended category $\mathcal{E}(\mathcal{C})$ *of $\mathcal{C}$ is defined as follows: An object of the category $\mathcal{E}(\mathcal{C})$ is defined as a morphism $(\alpha : A' \to A)$ in the original category $\mathcal{C}$. Given a pair of objects $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ of $\mathcal{E}(\mathcal{C})$, a morphism $\mathcal{X} \to \mathcal{Y}$ in category $\mathcal{E}(\mathcal{C})$ is an equivalence class of morphisms $f : A \to B$ of category $\mathcal{C}$ such that $f \circ \alpha = \beta \circ f'$ for some morphism $f' : A' \to B'$ in $\mathcal{C}$. Two morphisms $f_1 : A \to B$ and $f_2 : A \to B$ of $\mathcal{C}$ represent* identical *morphisms $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{E}(\mathcal{C})$ iff $f_1 - f_2 = \beta \circ F$ for some morphism $F : A \to B'$ of category $\mathcal{C}$. The morphism $\mathcal{X} \to \mathcal{Y}$, represented by $f : A \to B$, is denoted by*

$$[f] : (\alpha : A' \to A) \ \to \ (\beta : B' \to B) \qquad \text{or by} \qquad [f] : \mathcal{X} \to \mathcal{Y}. \tag{4.1}$$

Composition of morphisms in $\mathcal{E}(\mathcal{C})$ is defined as composition of the corresponding morphisms f in the category $\mathcal{C}$.

The category $\mathcal{E}(\mathcal{C})$ is an abelian category, cf. [13, Proposition 1.7].

It is shown in [13, Section 1.4], that any object $\mathcal{X}$ of $\mathcal{E}(\mathcal{C})$ is isomorphic in $\mathcal{E}(\mathcal{C})$ to an object $(\alpha : A' \to A)$, where the morphism α is injective.

4.2. **Projective and torsion objects.** There is a full embedding of $\mathcal{C}$ in $\mathcal{E}(\mathcal{C})$ which takes an object A of $\mathcal{C}$ to the zero morphism $(0 \to A)$ of $\mathcal{E}(\mathcal{C})$ and a $\mathcal{C}$-morphisms $f : A \to B$ to the morphism $[f] : (0 \to A) \to (0 \to B)$ of $\mathcal{E}(\mathcal{C})$. An object of $\mathcal{E}(\mathcal{C})$ is projective if and only if it is isomorphic to a Hilbertian module, i.e. to an object of $\mathcal{C}$, cf. [13, Proposition 1.9].

Given an object $\mathcal{X} = (\alpha : A' \to A)$ in the extended category $\mathcal{E}(\mathcal{C})$, its *projective part* is defined as the following object of $\mathcal{C}$

$$\mathcal{P}(\mathcal{X}) \ := \ A/\operatorname{cl}(\operatorname{im}(\alpha)). \tag{4.2}$$

Clearly, any morphism $[f] : \mathcal{X} \to \mathcal{Y}$ of $\mathcal{E}(\mathcal{C})$ induces a morphism $\mathcal{P}(f) : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$ between the projective parts. Thus we have a well defined functor

$$\mathcal{P} : \mathcal{E}(\mathcal{C}) \ \to \ \mathcal{C}. \tag{4.3}$$

An object $\mathcal{X} = (\alpha : A' \to A)$ of the extended category $\mathcal{E}(\mathcal{C})$ is called *torsion* iff the image of α is dense in A. One denotes by $\mathcal{T}(\mathcal{C})$ the full subcategory of $\mathcal{E}(\mathcal{C})$ generated by all

torsion objects. $\mathcal{T}(\mathcal{C})$ is called *the torsion subcategory of* $\mathcal{E}(\mathcal{C})$. It is shown in [13, §3], that: *if* $\mathcal{C}$ *is a finite von Neumann category, then* $\mathcal{T}(\mathcal{C})$ *is an abelian category.*

Objects of $\mathcal{T}(\mathcal{C})$ are called *torsion Hilbertian modules.*

There is a functor

$$\mathcal{T} : \mathcal{E}(\mathcal{C}) \ \to \ \mathcal{T}(\mathcal{C}), \qquad \mathcal{T} : (\alpha : A' \to A) \ \mapsto \ (\alpha : A' \to \mathrm{cl}(\mathrm{im}(\alpha))). \qquad (4.4)$$

Given $\mathcal{X} \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}))$, the object $\mathcal{T}(\mathcal{X})$ is called *the torsion part of* $\mathcal{X}$.

It is shown in [13, §3.4] that *every object* $\mathcal{X} \in \mathcal{E}(\mathcal{C})$ *is a direct sum of its torsion and projective parts,*

$$\mathcal{X} \ = \ \mathcal{T}(\mathcal{X}) \oplus \mathcal{P}(\mathcal{X}). \qquad (4.5)$$

4.3. **Isomorphisms in the extended category.** Let $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ be two objects of the extended category $\mathcal{E}(\mathcal{C})$. From Definition 4.1 we see that a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ induces an isomorphism in $\mathcal{E}(\mathcal{C})$ if and only if there exist $\mathcal{C}$-morphisms $\beta, f, f', g, g', F, G$ such that the following diagrams commute

$$
\begin{array}{ccc}
A' \xrightarrow{\alpha} A & \quad A' \xrightarrow{\alpha} A & \quad A' \xrightarrow{\alpha} A & \quad B' \xrightarrow{\beta} B \\
f' \downarrow \quad \downarrow f & \quad g' \uparrow \quad \uparrow g & \quad {}^{F}\nearrow \downarrow \mathrm{id}_A - gf & \quad {}^{G}\nearrow \downarrow \mathrm{id}_B - fg \quad (4.6) \\
B' \xrightarrow{\beta} B & \quad B' \xrightarrow{\beta} B & \quad A' \xrightarrow{\alpha} A & \quad B' \xrightarrow{\beta} B
\end{array}
$$

Lemma 4.4. *Let* $\mathcal{X} = (\alpha : A' \to A)$ *and* $\mathcal{Y} = (\beta : B' \to B)$ *be two objects of* $\mathcal{E}(\mathcal{C})$ *which are isomorphic is* $\mathcal{E}(\mathcal{C})$ *and such that the morphisms* α *and* β *are injective. Let* $f : A \to B$ *be a morphism in* $\mathcal{C}$ *which induces an isomorphism* $[f] : \mathcal{X} \to \mathcal{Y}$. *Let* $f' : A' \to B'$ *be as in the left diagram of* (4.6). *Then the following sequence of morphisms of* $\mathcal{C}$ *is exact*[4]*:*

$$0 \to \ A' \ \xrightarrow{(f', \alpha)} \ B' \oplus A \ \xrightarrow{\beta - f} \ B \ \to \ 0. \qquad (4.7)$$

Proof. We shall apply Proposition 1.6 from [13] which describes explicitly the kernels and cokernels in $\mathcal{E}(\mathcal{C})$. Clearly, $[f] : \mathcal{X} \to \mathcal{Y}$ being an isomorphism is equivalent to the vanishing of its kernel and cokernel in $\mathcal{E}(\mathcal{C})$. The cokernel of $[f]$ in $\mathcal{E}(\mathcal{C})$ equals $((\beta, -f) : B' \oplus A \to B)$ and its vanishing means that the $\mathcal{C}$-morphism $(\beta, -f) : B' \oplus A \to B$ is onto; this is equivalent to the exactness of (4.7) at the last term. The kernel of $[f]$ in $\mathcal{E}(\mathcal{C})$ equals $((f', \alpha) : A' \to P)$ where P denotes the kernel of $\beta + f : B' \oplus A \to B$ (here we use the assumption that β is injective). Hence vanishing of the kernel of $[f]$ is equivalent to the exactness of (4.7) in the middle term. The exactness of (4.7) in the first term follows from the injectivity of α. $\square$

Remark 4.5. One may think about an object $\mathcal{X} = (\alpha : A' \to A)$ of $\mathcal{E}(\mathcal{C})$ as of a short chain complex in $\mathcal{C}$ where α is the differential. If α is injective then this chain complex has

---

[4]Since the morphisms α and β are injective the morphism f' and the sequence (4.7) are completely determined by f.

cohomology in a single dimension (i.e. it is a resolution). A morphism in $\mathcal{E}(\mathcal{C})$

$$
\begin{array}{ccc}
A' & \xrightarrow{\ \alpha\ } & A \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
B' & \xrightarrow[\ \ \beta\ \ ]{} & B
\end{array}
\tag{4.8}
$$

is then a morphism of chain complexes. The chain complex (4.7) is the cone of this chain map. Hence the lemma above can be rephrased by saying that a morphism of the extended category is an isomorphism if and only if its cone is acyclic.

5. Determinant lines of objects in the extended category

Throughout this section $\mathcal{C}$ denotes a Hilbertian von Neumann category endowed with a finite trace, cf. Subsection 2.6. We denote by $\mathcal{E}(\mathcal{C})$ the extended category of $\mathcal{C}$, cf. Section 4.

Definition 5.1. *Let* $\mathcal{X} = (\alpha : A' \to A) \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}))$. *The determinant line of* $\mathcal{X}$ *is*

$$
\det \mathcal{X} \ := \det A \otimes (\det A' / \ker(\alpha))^*.
\tag{5.1}
$$

An element of the determinant line $\det \mathcal{X}$ can be represented by a symbol of the form

$$
\lambda \cdot \frac{\langle\,,\,\rangle}{\langle\,,\,\rangle'},
\tag{5.2}
$$

where $\lambda \in \mathbb{R}$ and $\langle\,,\,\rangle$, $\langle\,,\,\rangle'$ are $\mathcal{C}$-admissible scalar products on A, and on $A'/\ker(\alpha)$ correspondingly. The equivalence relation between symbols (5.2) follows from (3.1) and from the usual rules of fractions.

If $\mathcal{X} = (0 \to A)$ is a projective object in $\mathcal{E}(\mathcal{C})$, cf. Subsection 4.2, then the determinant line $\det \mathcal{X}$ coincides with $\det A$.

Let $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ be two objects of the extended category $\mathcal{E}(\mathcal{C})$ such that the morphisms α and β are injective. Consider a morphism $[f] : \mathcal{X} \to \mathcal{Y}$ represented by diagram (4.8). Assuming that $[f]$ is an isomorphism in $\mathcal{E}(\mathcal{C})$ we define the push-forward map $[f]_* : \det \mathcal{X} \to \det \mathcal{Y}$ associated with $[f]$ as follows. Fix arbitrary $\mathcal{C}$-admissible scalar products $\langle\,,\,\rangle_A$, $\langle\,,\,\rangle_{A'}$, $\langle\,,\,\rangle_{B'}$ on A, A' and B' correspondingly. Consider the exact sequence (4.7). The orthogonal complement to the image of (f', α) (with respect to the scalar product $\langle\,,\,\rangle_{B'} \oplus \langle\,,\,\rangle_A$) is mapped by $\beta - f$ isomorphically onto B. This isomorphism together with the restriction of the scalar product $\langle\,,\,\rangle_{B'} \oplus \langle\,,\,\rangle_A$ onto $\mathrm{im}(f', \alpha)^\perp$ determine a $\mathcal{C}$-admissible scalar product $\langle\,,\,\rangle_B$ on B. We set

$$
[f]_* \left(\frac{\langle\,,\,\rangle_A}{\langle\,,\,\rangle_{A'}} \right) = \frac{\langle\,,\,\rangle_B}{\langle\,,\,\rangle_{B'}} \in \det \mathcal{Y}.
$$

Recall that the symbol $\dfrac{\langle\,,\,\rangle_A}{\langle\,,\,\rangle_{A'}}$ denotes a nonzero element of the line $\det \mathcal{X}$.

Lemma 5.2. *The push-forward map* $[f]_* : \det \mathcal{X} \to \det \mathcal{Y}$ *is well-defined. It depends only on the class* $[f]$ *in* $\mathrm{Hom}_{\mathcal{E}(\mathcal{C})}(\mathcal{X}, \mathcal{Y})$.

Proof. Assume that $\widetilde{f} : A \to B$ is another morphism such that $[\widetilde{f}] = [f]$. Then there exists a morphism $R : A \to B'$ such that $\beta \circ R = f - \widetilde{f}$. Let $\widetilde{f'} : A' \to B'$ satisfy $\beta \circ \widetilde{f'} = \widetilde{f} \circ \alpha$. Then

$$\beta \circ (f' - \widetilde{f'}) \;=\; (f - \widetilde{f}) \circ \alpha \;=\; \beta \circ R \circ \alpha,$$

and, since β is injective, $f' - \widetilde{f'} = R \circ \alpha$.

Set

$$h \;=\; \begin{pmatrix} \mathrm{id}_{B'} & R \\ 0 & \mathrm{id}_A \end{pmatrix} : B' \oplus A \;\to\; B' \oplus A.$$

Then $h^{-1} = \begin{pmatrix} \mathrm{id}_{B'} & -R \\ 0 & \mathrm{id}_A \end{pmatrix} : B' \oplus A \to B' \oplus A$ and

$$(\widetilde{f'}, -\alpha) \;=\; (f' - R \circ \alpha, -\alpha) \;=\; h \circ (\widetilde{f'}, -\alpha),$$
$$\beta + \widetilde{f} \;=\; \beta + f - \beta \circ R \;=\; (\beta + \widetilde{f}) \circ h^{-1}.$$

Since $\mathrm{Det}_\tau(h) = \mathrm{Det}_\tau(h^*) = 1$ by (2.13) our claim follows now from Lemma 3.8. $\square$

To define the push-forward map $[f]_* : \det \mathcal{X} \to \det \mathcal{Y}$ without assuming that α and β are injective one observes that the determinant lines of $\mathcal{X}$ and of $\mathcal{X}' = (\alpha : A'/\ker\alpha \to A)$ are identical and similarly the determinant line of $\mathcal{Y}$ and of $\mathcal{Y}' = (\beta : B'/\ker\beta \to B)$ are identical. Any isomorphism $[f] : \mathcal{X} \to \mathcal{Y}$ given by the diagram (4.8) determines an isomorphism $[f] : \mathcal{X}' \to \mathcal{Y}'$ given by

$$
\begin{array}{ccc}
A'/\ker\alpha & \xrightarrow{\;\;\alpha\;\;} & A \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
B'/\ker\beta & \xrightarrow[\;\;\beta\;\;]{} & B.
\end{array}
\tag{5.3}
$$

The next claim follows directly from the definitions:

Proposition 5.3. *Let $[f] : \mathcal{X} \to \mathcal{Y}$ and $[g] : \mathcal{Y} \to \mathcal{Z}$ be isomorphisms. Then*

$$[g]_* \circ [f]_* \;=\; \big([g] \circ [f] \big)_*.$$

5.4. **Determinant line of a direct sum.** Given two objects $\mathcal{X}, \mathcal{Y} \in \mathcal{E}(\mathcal{C})$ there is a canonical isomorphism

$$\phi : \det \mathcal{X} \otimes \det \mathcal{Y} \to \det(\mathcal{X} \oplus \mathcal{Y}),
\tag{5.4}$$

compare (3.2).

In particular one obtains that $\det \mathcal{X}$ is canonically isomorphic to the product of the determinants of the projective and the torsion parts: $\det \mathcal{X} \simeq \det \mathcal{P}(\mathcal{X}) \otimes \det \mathcal{T}(\mathcal{X})$.

Lemma 5.5. *For any morphism $[f] : \mathcal{X} \to \mathcal{Y}$ of $\mathcal{E}(\mathcal{C})$ there is a canonical isomorphism*

$$\phi_f : \det \mathcal{Y} \otimes (\det \mathcal{X})^* \;\xrightarrow{\;\;\sim\;\;}\; \det \mathrm{coker}([f]) \otimes (\det \ker([f]))^*.$$

Proof. Let $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$. Without loss of generality we can assume that α and β are injective.

Let $f : A \to B$ be a representative of $[f]$. Then there is a unique $f' : A' \to B'$ such that $f \circ \alpha = \beta \circ f'$. Recall that the kernel and the cokernel of $[f]$ are described in Proposition 1.6 of [13] as follows.

$$\mathrm{coker}([f]) \;=\; ((\beta, -f) : B' \oplus A \to B), \qquad \ker([f]) \;=\; (i : P' \to P), \qquad (5.5)$$

where

$$P \;=\; \ker((\beta, -f) : B' \oplus A \to B), \qquad P' \;=\; \ker((\beta, -f \circ \alpha) : B' \oplus A' \to B), \qquad (5.6)$$

and $i : P' \to P$ is the restriction of the map $\mathrm{id} \oplus \alpha$ to P'. The assumption that β is injective allows to simplify the description of P'. Indeed, by (5.6), P' consists of those pairs $(a', b') \in A' \oplus B'$ such that $0 \;=\; f(\alpha(a')) - \beta(b') \;=\; \beta(f'(a') - b'))$; the latter is equivalent to $f'(a') = b'$. Thus there is a natural isomorphism $A' \xrightarrow{\sim} P'$ given by $a' \mapsto (f'(b'), a')$, and the kernel of $[f]$ can be rewritten as $\ker([f]) \;=\; (\iota : A' \to P)$, $\iota : a' \mapsto (f'(a'), \alpha(a'))$. Since α is injective so is ι. Thus we obtain

$$\det \ker([f]) \;=\; \det P \otimes (\det A')^* \;=\; \det(\ker(\beta \oplus -f)) \otimes (\det A')^*. \qquad (5.7)$$

From (5.5) we obtain

$$\det \mathrm{coker}([f]) \;=\; \det B \otimes (\det(B' \oplus A / \ker(\beta \oplus -f)))^* \;=$$
$$=\; \det B \otimes (\det B')^* \otimes (\det A)^* \otimes \det(\ker(f \oplus -\beta)). \qquad (5.8)$$

Combining (5.5), (5.7) and (5.8) we obtain a natural isomorphism

$$\det \mathrm{coker}([f]) \otimes (\det \ker([f]))^* \;\cong\; \det B \otimes (\det B')^* \otimes (\det A)^* \otimes \det A'. \qquad (5.9)$$

Since the RHS of (5.9) is exactly $\det \mathcal{Y} \otimes (\det \mathcal{X})^*$ the lemma follows. $\qquad \square$

6. τ-TRIVIAL TORSION OBJECTS AND THEIR DETERMINANT LINES

Let $\mathcal{C}$ be a Hilbertian von Neumann category endowed with a finite trace τ. We have shown above how one may associate determinant lines to the objects of the extended category $\mathcal{E}(\mathcal{C})$.

The trivial object of $\mathcal{E}(\mathcal{C})$ is represented by any morphism $\mathcal{X} = (\alpha : A' \to A)$ such that α is onto. Let us show that the determinant line of the trivial object is *trivial* in the sense that it contains a canonical nonzero element. The choice of such element allows to establish a unique isomorphism to standard line $\mathbb{R}$ sending the canonical element to 1. Let $\langle\,,\,\rangle$ be a $\mathcal{C}$-admissible scalar product on A. The morphism $\alpha : A' \to A$ determines an isomorphism $\alpha' : A'/\ker\alpha \to A$. We set $\langle x, y \rangle' = \langle \alpha'(x), \alpha'(y) \rangle$ where $x, y \in A'/\ker\alpha$. The bracket $\langle\,,\,\rangle'$ is a $\mathcal{C}$-admissible scalar product on A'. It is easy to see that the following nonzero element

$$\frac{\langle\,,\,\rangle}{\langle\,,\,\rangle'} \;\in\; \det \mathcal{X}$$

is *canonical*, i.e. it does not depend on the choice of the initial scalar product $\langle\,,\,\rangle$.

The purpose of this section is to show that such trivialization of the determinant lines happens for a class of torsion objects which we call τ-*trivial*; the latter class depends on the choice of the trace τ.

Let $\mathcal{X} = (\alpha : A' \to A)$ be a torsion object of $\mathcal{E}(\mathcal{C})$. This means that the image of α is dense in A. Without loss of generality we may assume that α is injective.

Definition 6.1. *A morphism $\alpha : A' \to A$ is called a τ-isomorphism if it is injective, has dense image, and the integral*

$$\int\limits_0^\infty \ln \lambda \, d\phi(\lambda) \; > \; -\infty \tag{6.1}$$

is finite.

Here $\phi(\lambda)$ denotes the spectral density function $\phi(\lambda) = \tau(E_\lambda)$ of the self-adjoint operator

$$(\alpha^*\alpha)^{1/2} = \int\limits_0^\infty \lambda \, dE_\lambda. \tag{6.2}$$

The adjoint α^* is calculated with respect to a choice of $\mathcal{C}$-admissible scalar products $\langle \, , \, \rangle$ on A and $\langle \, , \, \rangle_1$ on A'. The finiteness of the integral (6.1) is independent of this choice (see Proof of Proposition 3.2 in [6]). Note that the integral (6.1) may only diverge at point $\lambda = 0$.

We shall use the following notation

$$\mathrm{Det}_\tau(\alpha^*\alpha) = \exp\left[2 \int\limits_0^\infty \ln(\lambda) \, d\phi(\lambda) \right] \tag{6.3}$$

where α and $\phi(\lambda)$ are as above. It is an extension of the Fuglede - Kadison determinant, see [17], Lemma 5; this notation is also compatible with the formula (2.14) above.

Definition 6.2. *A torsion object $\mathcal{X}$ is called τ-trivial if it is isomorphic to $(\alpha : A' \to A)$ where α is a τ-isomorphism.*

Lemma 6.3. *The determinant line $\det \mathcal{X}$ of any τ-trivial torsion object $\mathcal{X}$ is trivialized in the above sense, i.e. it contains a canonical nonzero element.*

Proof. Let $\mathcal{X} = (\alpha : A' \to A)$ with α being a τ-isomorphism. Fix $\mathcal{C}$-admissible scalar products $\langle \, , \, \rangle$ on A and $\langle \, , \, \rangle'$ on A'. Consider the spectral decomposition (6.2) where the adjoint α^* is calculated with respect to the chosen scalar products. Let us show that the following nonzero element

$$\frac{\langle \, , \, \rangle}{\langle \, , \, \rangle'} \cdot \sqrt{\mathrm{Det}_\tau(\alpha^*\alpha)} \; \in \; \det \mathcal{X} \tag{6.4}$$

is independent of the choices of the scalar products $\langle \, , \, \rangle$ and $\langle \, , \, \rangle'$. Let $\beta : A \to A$ and $\gamma : A' \to A'$ be positive self-adjoint invertible $\mathcal{C}$-morphisms. Consider the new scalar

products $\langle\,,\,\rangle_1 = \langle\beta\cdot,\cdot\rangle$ and $\langle\,,\,\rangle_2 = \langle\gamma\cdot,\cdot\rangle'$ on A and on A' correspondingly. The adjoint of α with respect to the new pair of scalar products is $\gamma^{-1}\alpha^*\beta$. Using the multiplicativity property of the Fuglede - Kadison determinant we find

$$\mathrm{Det}_\tau(\gamma^{-1}\alpha^*\beta\alpha)^{1/2} = \mathrm{Det}_\tau(\gamma)^{-1/2}\,\mathrm{Det}_\tau(\alpha^*\beta\alpha)^{1/2} =$$
$$= \mathrm{Det}_\tau(\gamma)^{-1/2}\,\mathrm{Det}_\tau(\alpha\alpha^*\beta)^{1/2} = \mathrm{Det}_\tau(\gamma)^{-1/2}\,\mathrm{Det}_\tau(\alpha\alpha^*)^{1/2}\,\mathrm{Det}_\tau(\beta)^{1/2} =$$
$$= \mathrm{Det}_\tau(\gamma)^{-1/2}\,\mathrm{Det}_\tau(\alpha^*\alpha)^{1/2}\,\mathrm{Det}_\tau(\beta)^{1/2}$$

Relation (3.1) gives $\langle\,,\,\rangle_1 = \mathrm{Det}_\tau(\beta)^{-1/2}\cdot\langle\,,\,\rangle$ and $\langle\,,\,\rangle_2 = \mathrm{Det}_\tau(\gamma)^{-1/2}\cdot\langle\,,\,\rangle'$. Hence combining the above relations we find that the new density

$$\frac{\langle\,,\,\rangle_1}{\langle\,,\,\rangle_2}\cdot\mathrm{Det}_\tau(\gamma^{-1}\alpha^*\beta\alpha)^{1/2} \tag{6.5}$$

equals (6.4). $\qquad\square$

7. Determinant line of a chain complex

7.1. Let $\mathcal{M} = \oplus\mathcal{M}^i$ be a graded object of $\mathcal{E}(\mathcal{C})$. The determinant line of $\mathcal{M}$ is defined as

$$\det\mathcal{M} := \otimes(\det\mathcal{M}^i)^{(-1)^i} \tag{7.1}$$

where $(\det\mathcal{M}^i)^{-1}$ denotes the dual line to $\det\mathcal{M}^i$.

Suppose now that

$$(\mathcal{M},\partial): \quad 0 \to \mathcal{M}^0 \xrightarrow{\ \partial_1\ } \mathcal{M}^1 \xrightarrow{\ \partial_2\ } \cdots \xrightarrow{\ \partial_n\ } \mathcal{M}^n \to 0 \tag{7.2}$$

is a chain complex in the abelian category $\mathcal{E}(\mathcal{C})$. We denote by

$$\mathcal{H}^*(\mathcal{M}) = \bigoplus_{i=0}^{n}\mathcal{H}^i(\mathcal{M}) \tag{7.3}$$

its cohomology. Even if the initial complex $\mathcal{M}$ consisted of projective objects (i.e. if $\mathcal{M}^i \in \mathrm{Ob}(\mathcal{C})$ for all i), the cohomology $\mathcal{H}^*(\mathcal{M})$ may have nontrivial torsion, thus it is well-defined only as an object of the extended category $\mathcal{E}(\mathcal{C})$, i.e. $\mathcal{H}^i(\mathcal{M}) \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}))$. It is called the *extended cohomology* of $\mathcal{M}$. The notion of extended cohomology was first introduced in [11], [12].

In the special case when the complex $\mathcal{M}$ is projective, the extended cohomology can explicitly be expressed by

$$\mathcal{H}^i(\mathcal{M}) = (\partial_i: \mathcal{M}^{i-1} \to \ker(\partial_{i+1})), \tag{7.4}$$

see [11], [12].

The projective part of the extended cohomology is then isomorphic to the *reduced cohomology*

$$H^i(\mathcal{M}) := \ker(\partial_{i+1})/\mathrm{cl}(\mathrm{im}(\partial_i)) \in \mathrm{Ob}(\mathcal{C}), \qquad i = 0,\ldots,n, \tag{7.5}$$

of $(\mathcal{M},\partial)$. The notion of reduced L^2-cohomology was originally introduced by M.Atiyah [1]. The torsion part of the extended cohomology is responsible for, so called, "zero in

the continuous spectrum phenomenon". In particular, it completely describes the Novikov-Shubin invariants of $(\mathcal{M}, \partial)$, cf. [13, §3]. The torsion part of $\mathcal{H}^i(\mathcal{M})$ equals

$$\mathcal{T}(H^i(\mathcal{M})) = (\partial_i : \mathcal{M}^{i-1} \to \mathrm{cl}(\mathrm{im}(\partial_i)))$$

see [12], [13].

Proposition 7.2. *The chain complex* (7.2) *defines a canonical isomorphism*

$$\nu_{\mathcal{M}} : \det \mathcal{M} \longrightarrow \det \mathcal{H}^*(\mathcal{M}), \tag{7.6}$$

preserving the orientations of the lines.

Proof. We use here the notions "image" and "kernel" in their categorical sense; we apply them to morphisms of the abelian category $\mathcal{E}(\mathcal{C})$.

Let $Z^i(\mathcal{M}) = \ker[\partial : \mathcal{M}^i \to \mathcal{M}^{i+1}]$ and $B^i(\mathcal{M}) = \mathrm{im}[\partial : \mathcal{M}^{i-1} \to \mathcal{M}^i]$ be the "cycles" and the boundaries, correspondingly. One obtains two familiar exact sequences in $\mathcal{E}(\mathcal{C})$:

$$0 \to B^i(\mathcal{M}) \to Z^i(\mathcal{M}) \to \mathcal{H}^i(\mathcal{M}) \to 0$$

and

$$0 \to Z^{i-1}(\mathcal{M}) \to \mathcal{M}^{i-1} \to B^i(\mathcal{M}) \to 0.$$

The isomorphisms of Proposition 3.7 applied twice (alternatively one may appeal here to Lemma 5.5), determines a canonical isomorphism

$$\det \mathcal{H}^i(\mathcal{M}) \simeq (\det \mathcal{M}^{i-1})^* \otimes \det Z^{i-1}(\mathcal{M}) \otimes \det Z^i(\mathcal{M}).$$

Now one obtains

$$\det \mathcal{H}^*(\mathcal{M}) = \prod_{i=0}^{n} (\det \mathcal{H}^i(\mathcal{M}))^{(-1)^i} =$$

$$= \prod_{i=0}^{n} (\det \mathcal{M}^i)^{(-1)^i} \otimes \prod_{i=0}^{n} \left[\det Z^i(\mathcal{M}) \otimes \det Z^{i-1}(\mathcal{M})\right]^{(-1)^i} =$$

$$= \det(\mathcal{M}) \otimes \prod_{i=0}^{n} \left[\det Z^i(\mathcal{M}) \otimes \det Z^{i-1}(\mathcal{M})\right]^{(-1)^i}.$$

The main point of the proof is the observation that the product of the square brackets in the last formula is a trivial line in the sense that it contains a canonical nonzero element (denoted 1) and so it is canonically isomorphic to $\mathbb{R}$. We see that the line $\det \mathcal{H}(\mathcal{M})$ is obtained from the line $\det \mathcal{M}$ by tensoring with a trivialized line containing a canonical element which we shall denote by 1. The canonical isomorphism (7.6) is defined by the formula

$$\nu_{\mathcal{M}}(x) = x \otimes 1.$$

$\square$

7.3. **Direct sum of complexes.** Let $(\mathcal{M}, \partial^{\mathcal{M}})$ and $(\mathcal{N}, \partial^{\mathcal{N}})$ be two complexes of objects in $\mathcal{E}(\mathcal{C})$. Consider their direct sum $(\mathcal{M} \oplus \mathcal{N}, \partial^{\mathcal{M}} \oplus \partial^{\mathcal{N}})$. It follows from (5.4) that

$$\nu_{\mathcal{M} \oplus \mathcal{N}} = \nu_{\mathcal{M}} \otimes \nu_{\mathcal{N}}. \tag{7.7}$$

7.4. **Torsion of a complex of projective objects.** Suppose

$$(C, \partial): \quad 0 \to C^0 \xrightarrow{\partial_1} C^1 \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_n} C^n \to 0 \tag{7.8}$$

is a complex in $\mathcal{C}$. We shall consider it as a complex of projective objects of $\mathcal{E}(\mathcal{C})$. Fix $\mathcal{C}$-admissible scalar products $\langle \, , \, \rangle_i$ on C^i. They determine elements $\sigma_i \in \det C^i$ and

$$\sigma = \prod_{i=0}^{n} \sigma_i^{(-1)^i} \in \det C$$

Definition 7.5. *The positive element*

$$\rho_C := \nu_C(\sigma) \in \det \mathcal{H}^*(C) \tag{7.9}$$

is called the torsion of the complex C.

Remark 7.6. The torsion ρ_C depends on the choice of the scalar products $\langle , \rangle_i$. In [6], the torsion was defined more invariantly as the element of the line $(\det C)^* \otimes \det \mathcal{H}^*(C)$ canonically determined by the map ν_C. In this paper we adopt a less invariant definition (7.9), since it is more consistent with the usual notions of combinatorial and de Rham torsions. Note that, using our notation, the torsion of [6] can be written as $\sigma^* \otimes \rho_C$, where σ^* is the unique element of $(\det C)^*$ such that $\langle \sigma^*, \sigma \rangle = 1$.

7.7. **Torsion of an acyclic complex of projective objects.** Assume now that the complex (7.8) is acyclic (by this we mean that $\operatorname{im} \partial_{i-1} = \ker \partial_i$. The acyclicity implies in particular that $\operatorname{im}(\partial_i)$ is a closed subspace of C^i). We now calculate the torsion $\rho_C = \nu_C(\sigma) \in \det \mathcal{H}(C) \simeq \mathbb{R}$ which is in this situation a positive real number.

Let $\partial_i^* : C^i \to C^{i-1}$ denote the adjoint of ∂_i with respect to the chosen scalar products and let

$$\Delta_i := \partial_{i+1}^* \partial_{i+1} + \partial_i \partial_i^* : C^i \to C^i$$

be the Laplacian. Since the complex (C, ∂) is acyclic, Δ_i is a positive invertible operator. We denote by $\operatorname{Det}_\tau(\Delta_i)$ the Fuglede-Kadison determinant of Δ_i, cf. Subsection 2.10.

Proposition 7.8. *Assume that the chain complex (7.8) is acyclic. Then its torsion is given by the formula*

$$\rho_C = \exp\left(\frac{1}{2} \sum_{i=0}^{n} (-1)^i i \log \operatorname{Det}_\tau(\Delta_i)\right). \tag{7.10}$$

Proof. Set

$$A^i := \ker(\partial_{i+1}) = \operatorname{im}(\partial_i); \qquad B^i := \ker(\partial_i^*) = \operatorname{im}(\partial_{i+1}^*) = \left(\ker(\partial_{i+1})\right)^\perp.$$

Then $C^i = A^i \oplus B^i$ and the complex (7.8) decomposes into direct sum of the following "short" chain complexes

$$(\mathcal{M}_i, \partial): \quad 0 \to \cdots 0 \to B^{i-1} \xrightarrow{\partial_i|_{B^{i-1}}} A^i \to 0 \to \cdots \to 0,$$

where $i = 1, , \ldots, , n$. The scalar products $\langle \, , \, \rangle_i$ on C^i $(i = 0, \ldots, n)$ induce scalar products on A^i and B^i and hence the elements $\mu_i \in \det \mathcal{M}_i$. Clearly,

$$\sigma \; = \; \mu_1 \otimes \cdots \otimes \mu_n, \qquad \nu_C \; = \; \nu_{\mathcal{M}_1} \otimes \cdots \otimes \nu_{\mathcal{M}_n}. \tag{7.11}$$

From (3.5) we see that

$$\rho_{\mathcal{M}_i} \; = \; \nu_{\mathcal{M}_i}(\mu_i) \; = \; (\mathrm{Det}_\tau(\partial_i^* \partial_i|_{B^{i-1}}))^{(-1)^i/2}. \tag{7.12}$$

Combining (7.11) and (7.12) with (7.9) we obtain

$$\rho_C \; = \; \mathrm{Det}_\tau(\partial_1^* \partial_1|_{B^0})^{-1/2} \cdot \mathrm{Det}_\tau(\partial_2^* \partial_2|_{B^1})^{1/2} \cdots \mathrm{Det}_\tau(\partial_n^* \partial_n|_{B^{n-1}})^{(-1)^n/2}$$

$$= \; \exp\Big(\frac{1}{2} \sum_{i=1}^n (-1)^i \log \mathrm{Det}_\tau\big(\partial_i^* \partial_i|_{B^{i-1}}\big)\Big). \tag{7.13}$$

It remains to show that the right hand sides of (7.10) and (7.13) coincide. The Laplacian has the following matrix form with respect to the decomposition $C^i = A^i \oplus B^i$:

$$\Delta_i \; = \; \begin{pmatrix} \partial_i \partial_i^* & 0 \\ 0 & \partial_{i+1}^* \partial_{i+1} \end{pmatrix}.$$

Hence, using (2.13), we obtain

$$\mathrm{Det}_\tau(\Delta_i) \; = \; \mathrm{Det}_\tau(\partial_i \partial_i^*|_{A^i}) \cdot \mathrm{Det}_\tau(\partial_{i+1}^* \partial_{i+1}|_{B^i}). \tag{7.14}$$

The restriction of ∂_i to B^{i-1} maps it isomorphically onto A^i and we have

$$\partial_i|_{B^{i-1}} \big(\partial_i^* \partial_i|_{B^{i-1}}\big) \; = \; \big(\partial_i \partial_i^*|_{A^i}\big) \partial_i|_{B^{i-1}}.$$

Hence,

$$\mathrm{Det}_\tau(\partial_i^* \partial_i|_{B^{i-1}}) \; = \; \mathrm{Det}_\tau(\partial_i \partial_i^*|_{A^i}) \tag{7.15}$$

by Theorem 2.11(a). Using (7.14) and (7.15) we get

$$\sum_{i=0}^n (-1)^i i \log \mathrm{Det}_\tau \Delta_i \; = \; \sum_{i=0}^{n-1} (-1)^i i \big(\log \mathrm{Det}_\tau(\partial_i^* \partial_i|_{B^{i-1}}) + \log \mathrm{Det}_\tau(\partial_{i+1}^* \partial_{i+1}|_{B^i})\big)$$

$$= \; \sum_{i=1}^n (-1)^i \log \mathrm{Det}_\tau(\partial_i^* \partial_i|_{B^{i-1}}). \tag{7.16}$$

Combining (7.13) and (7.16) we obtain (7.10). $\qquad\qquad \square$

8. TORSION AND EXACT SEQUENCES

In this section we study the relationship between the torsions of the terms of a short exact sequence of complexes. Roughly speaking the main result states that the torsion of the middle term equals to the product of the torsions of the two other terms. This result will be used in Section 10 to show that the combinatorial torsion of a flat Hilbertian bundle is invariant under subdivisions. It will be also used in Section 13 to prove that the ratio of the de Rham and the combinatorial torsions equals the relative torsion introduced in [8].

8.1. **The setting.** Let $\mathcal{C}$ be a Hilbertian von Neumann category endowed with a finite trace τ. Let

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0. \tag{8.1}$$

be an exact sequence of chain complexes (L, ∂^L), (M, ∂^M) and (N, ∂^N) of objects of $\mathcal{C}$. Let

$$\psi : \det M \xrightarrow{\sim} \det L \otimes \det N$$

be the isomorphism determined by (8.1), cf. Proposition 3.7.

Choose an identification of M, as a graded Hilbertian module, with the direct sum of L and N. With respect to this identification we have the following matrix representations

$$\alpha = \begin{pmatrix} \mathrm{id}_L \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0, & \mathrm{id}_N \end{pmatrix}, \quad \partial^M = \begin{pmatrix} \partial^L & f \\ 0 & \partial^N \end{pmatrix}$$

where $f : N^* \to L^{*+1}$ is a morphism satisfying

$$f \circ \partial^N = -\partial^L \circ f.$$

f induces a morphism of the extended cohomology $f_* : \mathcal{H}^*(N) \longrightarrow \mathcal{H}^{*+1}(L)$. The long exact sequence of cohomology corresponding to (8.1) has the form

$$\cdots \xrightarrow{f_*} \mathcal{H}^i(L) \xrightarrow{\alpha_*} \mathcal{H}^i(M) \xrightarrow{\beta_*} \mathcal{H}^i(N) \xrightarrow{f_*} \mathcal{H}^{i+1}(L) \xrightarrow{\alpha_*} \cdots \tag{8.2}$$

Using this exact sequence one defines a natural isomorphism

$$\delta : \det \mathcal{H}(L) \otimes \det \mathcal{H}(N) \xrightarrow{\sim} \det \mathcal{H}(M). \tag{8.3}$$

Let k_i and c_i denote the kernel and cokernel of the morphism $f_* : \mathcal{H}^{i-1}(N) \to \mathcal{H}^i(L)$, correspondingly. Lemma 5.5 gives a natural isomorphism

$$\det \mathcal{H}^i(L) \otimes (\det \mathcal{H}^{i-1}(N))^* \xrightarrow{\sim} \det c_i \otimes (\det k_i)^*. \tag{8.4}$$

The isomorphism of Proposition 3.7 applied to the exact sequence[5]

$$0 \to c_i \to \mathcal{H}^i(M) \to k_{i+1} \to 0$$

gives a natural isomorphism

$$\det c_i \otimes \det k_{i+1} \xrightarrow{\sim} \det \mathcal{H}^i(M). \tag{8.5}$$

---

[5]Formally Proposition 3.7 deals with objects of the initial category $\mathcal{C}$. However a similar statement with $M, M', M" \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}))$ is true as follows from Lemma 5.5

The alternating product of isomorphisms (8.4) and (8.5) produce the isomorphisms

$$\det \mathcal{H}(L) \otimes \det \mathcal{H}(N) \to \det c \otimes (\det k)^* \to \det \mathcal{H}(M)$$

and their composition is denoted by δ, see (8.3).

The following statement is the main result of this section.

Theorem 8.2. *Under the above conditions one has*

$$\delta \circ (\nu_L \otimes \nu_N) \circ \psi \ = \ \nu_M : \ \det M \to \det \mathcal{H}(M), \tag{8.6}$$

where ν_L, ν_M, and ν_N are the canonical isomorphisms of the determinant lines defined in Proposition 7.2 and ψ is the isomorphism of Proposition 3.7.

Proof. This statement is similar to Theorem 3.2 of Milnor [28] stating that the torsion of an extension equals the product of the individual torsions times the torsion of the exact homological sequence. The full proof of Theorem 8.6 is quite long and technical; therefore we have decided to publish it elsewhere. In the sequel this theorem is used only once (in the proof of Theorem 13.5) and only in a special case when at least one of the complexes L, N, M is acyclic. In this special case Theorem 8.2 admits a simpler proof which is still quite involved. We postpone it to a further publication. □

8.3. **The relationship between the torsions.** We keep the notations introduced above. Fix $\mathcal{C}$-admissible scalar products on L and N. They determine a $\mathcal{C}$-admissible scalar product on M. Although this scalar product on M is not unique, its class in $\det M$ depends only on the classes in $\det L$ and $\det N$ represented by the given scalar products on L and on N. Let $\rho_L \in \det \mathcal{H}^*(L)$, $\rho_M \in \det \mathcal{H}^*(M)$, and $\rho_N \in \det \mathcal{H}^*(N)$ be the torsions of the complexes (L, ∂^L), (M, ∂^M) and (N, ∂^N) respectively, cf. Definition 7.5. Let $\sigma_L \in \det L$, $\sigma_M \in \det M$, and $\sigma_N \in \det N$ be the elements induced by the scalar products, cf. Subsection 7.4. By definition, $\psi_{\alpha,\beta}(\sigma_M) \ = \ \sigma_L \otimes \sigma_N$. Hence we obain:

Theorem 8.4. $\delta(\rho_L \otimes \rho_N) = \rho_M$.

8.5. **Torsion of the cone complex.** The above results can be reformulated in terms of the cone complex. Suppose (C, ∂) and $(\widetilde{C}, \tilde{\partial})$ are chain complexes in $\mathcal{C}$. Let $f : C \to \widetilde{C}$ be a chain morphism. Consider the cone of f, i.e., the chain complex $\mathrm{Cone}(f)$:

$$0 \ \to \ C^0 \oplus 0 \ \xrightarrow{D_1} \ C^1 \oplus \widetilde{C}^0 \ \xrightarrow{D_2} \ \cdots \ \xrightarrow{D_n} \ C^{n-1} \oplus \widetilde{C}^n \ \xrightarrow{D_{n+1}} \ 0 \oplus \widetilde{C}^n \ \to \ 0, \tag{8.7}$$

where

$$D_i \ = \ \begin{pmatrix} -\partial_i & 0 \\ f_{i-1} & \tilde{\partial}_{i-1} \end{pmatrix}. \tag{8.8}$$

Fix $\mathcal{C}$-admissible scalar products on C and $\widetilde{C}$. They induce a $\mathcal{C}$-admissible scalar product on $\mathrm{Cone}(f)$ in an obvious way. Let $\rho_C \in \det \mathcal{H}(C)$, $\rho_{\widetilde{C}} \in \det \mathcal{H}(\widetilde{C})$ and $\rho_f \in \det \mathcal{H}(\mathrm{Cone}(f))$ be the torsions of the complexes (C, ∂), $(\widetilde{C}, \tilde{\partial})$ and $\mathrm{Cone}(f)$ respectively, defined using these scalar products, cf. Definition 7.5.

From Theorem 8.4 we obtain:

Corollary 8.6. *Using the notation introduced above we have*

$$\rho_f \;=\; \delta_f \left(\rho_C \otimes (\rho_{\widetilde{C}})^* \right), \tag{8.9}$$

where $(\rho_{\widetilde{C}})^* \in (\det(\mathcal{H}(\widetilde{C})))^*$ denotes the dual of $\rho_{\widetilde{C}}$.

This statement is equivalent to Theorem 8.4. The dual $(\rho_{\widetilde{C}})^*$ of the torsion appears because of the grading shift.

9. DETERMINANT CLASS CONDITION FOR CHAIN COMPLEXES

In this section we recall the determinant class condition for chain complexes in Hilbertian categories. Historically, this condition appeared as a necessary requirement for the L^2-torsion to be well defined. We show that the determinant class condition for a chain complex C is equivalent to the τ-triviality of the torsion parts of the extended L^2-cohomology $\mathcal{H}(C)$. As we have shown in §6, the τ-triviality implies that the determinant lines of the torsion parts of the extended cohomology could be canonically trivialized. Hence in this case the determinant line $\det \mathcal{H}(C)$ can be canonically identified with the determinant line $\det H(C)$ where $H(C)$ is the reduced L^2-cohomology of C. This explains why under the determinant class condition for a chain complex C the torsion is well defined as an element of $\det(H(C))$, as was proven in [6].

Definition 9.1. *Let $\mathcal{C}$ be a Hilbertian von Neumann category supplied with a non-negative, normal, and faithful trace τ. A chain complex*

$$(C, \partial): \quad 0 \;\to\; C^0 \;\xrightarrow{\partial_1}\; C^1 \;\xrightarrow{\partial_2}\; \cdots \;\xrightarrow{\partial_{n-1}}\; C^n \;\to\; 0 \tag{9.1}$$

in $\mathcal{C}$ is said to be of τ-determinant class if the induced maps

$$\partial_j : C^{j-1}/\ker(\partial_j) \;\to\; \mathrm{cl}(\mathrm{im}(\partial_j)) \tag{9.2}$$

are τ-isomorphisms for all $j = 0, \ldots, n$.

Compare [5] and [6].

Recall that the extended cohomology $\mathcal{H}^*(C)$ of the complex (9.1) is defined by formula (7.4) and the reduced cohomology is defined by formula (7.5).

Lemma 9.2. *A chain complex C as above is of τ-determinant class if and only iff the torsion parts $\mathcal{T}(\mathcal{H}^i(C))$ of the extended cohomology are τ-trivial for all i.*

Proof. The torsion part $\mathcal{T}(\mathcal{H}^i(C))$ equals $(\partial_j : C^{j-1}/\ker(\partial_j) \to \mathrm{cl}(\mathrm{im}(\partial_j)))$, see [12]. Hence the result follows by comparing Definition 9.1 and Definition 6.2.

$\square$

Corollary 9.3. *Given a chain complex (9.1) of determinant class, there is a canonical orientation preserving isomorphism between the determinant lines of the extended and the reduced cohomology of (C, ∂):*

$$\det \mathcal{H}^j(C) \;\xrightarrow{\sim}\; \det H^j(C, \partial), \quad j = 0, \ldots, n.$$

Proof. Recall from [12] that the reduced cohomology is naturally isomorphic to the projective part of the extended cohomology. Hence, in using (4.5) and (5.4), we see that

$$\det \mathcal{H}^i(C) \simeq \det H^i(C) \otimes \det \mathcal{T}(\mathcal{H}^i(C)) \simeq \det H^i(C)$$

assuming that C is of τ-determinant class. On the last stage we have applied Lemma 9.2 and Lemma 6.3 using the fact that the determinant line $\det \mathcal{T}(\mathcal{H}^i(C))$ of the torsion part is canonically trivialized.   □

From Propositions 7.2 and 9.3 we obtain the following:

Corollary 9.4. *Given a chain complex* (9.1) *of determinant class, there is a canonical orientation preserving isomorphism*

$$\widetilde{\phi}_C : \det C \ \xrightarrow{\sim} \ \det H^*(C, \partial).$$

9.5. **The torsion of a determinant class complex.** Using Corollary 9.3 we can view the torsion ρ_C of a determinant class complex (C, ∂) as a positive element of the determinant line $\det(H^*(C, \partial))$ of the reduced cohomology. In this case our notion of torsion essentially reduces to the one considered in [6]. More precisely, if the complex (C, ∂) is of determinant class, then the element

$$\sigma^* \otimes \psi_C(\rho_C) \ \in \ (\det C)^* \otimes \det(H^*(C, \partial))$$

coincides with the torsion defined in [6].

10. COMBINATORIAL L^2-TORSION WITHOUT THE DETERMINANT CLASS CONDITION

In this section we suggest a generalization of the classical construction of the combinatorial torsion of Reidemeister, Franz and de Rham to the case of infinite dimensional representations. Our torsion represents a nonzero element of the determinant line of the *extended L^2* cohomology. Intuitively, the torsion is a density supported on the extended L^2 cohomology of the polyhedron.

In more detail, given a finite polyhedron K and a unimodular representation of its fundamental group on a Hilbertian module M, the torsion invariant defined in this section is a positive element of the determinant line $\det \mathcal{H}^*(K; M)$. Here $\mathcal{H}^*(K, M)$ denotes the extended L^2-homology of K with coefficients in M, as defined originally in [13], §6.5. The definition of $\mathcal{H}^*(K; M)$ is briefly repeated below.

The main advantage of the construction suggested here is that it requires no additional assumptions. In particular, unlike the work of the previous authors studying the construction of L^2 torsion, we do not require our complexes be of determinant class.

We show that under the determinant class assumption the torsion defined in this section coincides with the torsion defined in our previous work [6] where the torsion was understood as an element of the determinant line of the *reduced L^2* cohomology.

10.1. **Hilbertian bimodule.** Let K be a finite cell complex. Denote by $\pi = \pi_1(K)$ its fundamental group and by $C_*(\widetilde{K})$ the cellular chain complex of the universal covering $\widetilde{K}$ of K. The group π acts on $C_*(\widetilde{K})$ from the left and $C_*(\widetilde{K})$ becomes a chain complex of free finitely generated $\mathbb{Z}[\pi]$-modules with lifts of the cells of K representing a free basis of $C_*(\widetilde{K})$ over $\mathbb{Z}[\pi]$.

Let M be a Hilbert space on which a von Neumann algebra $\mathcal{A}$ of bounded linear operators acts from the right. As in §2.5.2 we will denote by $\mathcal{B}(M) = \mathcal{B}_{\mathcal{A}}(M)$ the commutant of M; recall that $\mathcal{B}_{\mathcal{A}}(M)$ is the ring of all bounded linear maps $M \to M$ commuting with the right $\mathcal{A}$-action.

We shall assume that the algebra $\mathcal{A}$ is supplied with a finite, normal and faithful trace τ.

Consider a linear representation of the group π in M. It is a group homomorphism $\pi \to \mathcal{B}(M) = \mathcal{B}_{\mathcal{A}}(M)$. We think of π as acting linearly on M from the left (via the representation $\pi \to \mathcal{B}(M)$); thus M obtains an $(\pi - \mathcal{A})$-bimodule structure. In this situation we say that M is an *Hilbertian $(\pi - \mathcal{A})$-bimodule*.

A Hilbertian $(\pi - \mathcal{A})$-bimodule M is called *unimodular* if for every element $g \in \pi$ the Fuglede-Kadison determinant $\mathrm{Det}_\tau(g)$ equals 1 where g is viewed as an invertible linear operator $M \to M$ given by the right multiplication by g.

10.2. **Flat Hilbertian bundle.** Any Hilbertian $(\pi - \mathcal{A})$-bimodule M determines a *flat Hilbertian bundle over K with fiber M*. Consider the Borel construction $\mathcal{E} = \widetilde{K} \times_\pi M$ together with the obvious projection map $\mathcal{E} \to K$; it has a canonical structure of a flat $\mathcal{A}$-Hilbertian bundle. Equivalently, it can be viewed as a locally free sheaf of Hilbertian $\mathcal{A}$-modules.

10.3. **Examples.** As the first example, consider the well-known situation when $\mathcal{A} = \mathcal{N}(\pi)$ is the von Neumann algebra of π and $M = \ell^2(\pi)$ is the completion of the group algebra of π with respect to the canonical trace on it. Here $\mathcal{A}$ acts on $M = \ell^2(\pi)$ from the right and the group π acts on M from the left. This $(\pi - \mathcal{A})$-bimodule $M = \ell^2(\pi)$ is unitary and hence it is unimodular, as well.

As a more general example consider the following. Let V be a finite dimensional unimodular representation of π. Let $M = V \otimes_{\mathbb{C}} \ell^2(\pi)$. Here the right action of $\mathcal{A} = \mathcal{N}(\pi)$ on M is the same as the action on $\ell^2(\pi)$ and the left action of π is the diagonal action: $g(x \otimes v) = gx \otimes gv$ for $x \in \ell^2(\pi)$, $v \in V$, and $g \in \pi$.

10.4. **Construction of the torsion.** Let M be a $(\pi - \mathcal{A})$-bimodule. Let

$$C^*(K; M) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{K}), M). \tag{10.1}$$

It is a cochain complex of Hilbertian spaces (having right $\mathcal{A}$-action) and the differentials are bounded linear maps commuting with the $\mathcal{A}$-action. More precisely we may view complex (10.1) as a cochain complex in the von Neumann category $\mathcal{C} = \mathcal{H}_{\mathcal{A}}$ of Hilbertian representations of $\mathcal{A}$, see §2.5.2.

The cohomology $\mathcal{H}^*(K; M)$ of (10.1) is an object of the extended abelian category $\mathcal{E}(\mathcal{C})$. It is called *the extended cohomology* of K with coefficients in M. It was introduced originally in [11], [12].

By Proposition 7.2 we obtain a natural isomorphism

$$\phi_{C^*(K;M)} : \det C^*(K; M) \xrightarrow{\;\sim\;} \det \mathcal{H}^*(K; M). \tag{10.2}$$

Assume that the $(\pi - \mathcal{A})$-bimodule M is unimodular. Then there is natural isomorphism

$$\psi_K : (\det M)^{\chi(K)} \xrightarrow{\;\sim\;} \det C^*(K; M), \tag{10.3}$$

defined as follows. For any cell $e \subset K$ fix a lifting $\tilde{e}$ of e in the universal covering. Then the cells $\tilde{e}$ form a free $\mathbb{Z}[\pi]$-basis of the complex $C_*(\widetilde{K})$ and therefore induce an isomorphism

$$C^*(K; M) \xrightarrow{\;\sim\;} \oplus_{e \in K} M \tag{10.4}$$

between $C^*(K; M)$ and the direct sum of copies of M, one for each cell of K. Thus, the determinant line $\det C^*(K; M)$ can be identified with $(\det M)^{\chi(K)}$ since the cells of odd dimension contribute negative factors of $\det M$ into the total determinant line.

We only have to show that the above identification does not depend on the choice of the liftings $\tilde{e}$. For any cell $e \subset K$ choose a group element $g_e \in \pi$. Then the cells $g_e \tilde{e}$ form another set of liftings. Consider the map $\oplus M \to \oplus M$ which is given by the diagonal matrix with g_e on the diagonal. The Fuglede-Kadison determinant of this map equals 1 (since the representation M is unimodular). Hence we see that the isomorphism (10.3) is canonical.

Definition 10.5. *Fix a positive element $\sigma \in \det M$. It defines an element $\overline{\sigma} = \sigma^{\chi(K)} \in (\det M)^{-\chi(K)}$. The combinatorial L^2 torsion is defined by the formula*

$$\rho_K := \phi_{C^*(K,M)}\big(\psi_K(\overline{\sigma})\big) \in \det \mathcal{H}^*(K; M). \tag{10.5}$$

In other words, the combinatorial L^2 torsion can be defined as follows: let $\langle \, , \, \rangle$ be an $\mathcal{C} = \mathcal{H}_\mathcal{A}$-admissible scalar product on M which represents σ, cf. Subsection 3.1. For each cell $e \in K$ fix a lifting $\widetilde{e} \in \widetilde{K}$. Then $\langle \, , \, \rangle$ induces an admissible scalar product on $C^*(K; M)$ via the isomorphism (10.4). The combinatorial torsion is the torsion of the cochain complex $C^*(K; M)$ associated to this scalar product, cf. Definition 7.5.

Remark 10.6. 1. In the case when $\chi(K) = 0$ the torsion ρ_K is clearly independent of the choice of σ. This is always the case if K is a closed manifold of odd dimension.

2. If $\mathcal{A} = \mathbb{C}$ we arrive at the classical definitions, cf. [26, 27, 28, 29, 2].

In the case when the cochain complex $C^*(K, M)$ is of determinant class, Definition 10.5 reduces to the case considered in [6], cf. Subsection 9.5. If, in addition, the reduced L^2-homology $H^*(K, M)$ vanishes, we can identify the determinant line $\det H^*(K, M)$ with $\mathbb{R}$ and so ρ_K is just a number. Under this assumption it was studied in [7, 21, 23].

3. Although our notation ρ_K for the torsion invariant does not involve explicitly the trace $\tau : \mathcal{A} \to \mathbb{C}$, the whole construction (including the Fuglede-Kadison determinants and the determinant lines) certainly depend on the choice of the trace τ, cf. Remark 4.5.3 of [6].

Recall that the classical Reidemeister-Franz torsion is not, in general, a homotopy invariant, so one cannot expect homotopy invariance from our torsion invariant. But *combinatorial invariance* holds in the following sense

Theorem 10.7 (**Combinatorial Invariance**). *Let K be a finite polyhedron and let K' be its subdivision. Suppose that M is a Hilbertian unimodular $(\pi - \mathcal{A})$-bimodule. Let*

$$\psi : \mathcal{H}^*(K'; M) \;\to\; \mathcal{H}^*(K; M) \tag{10.6}$$

be the isomorphism induced on the extended L^2 cohomology by the subdivision chain map. Then the push-forward map

$$\psi_* : \det \mathcal{H}^*(K'; M) \;\to\; \det \mathcal{H}^*(K; M) \tag{10.7}$$

preserves the torsions, i.e. it maps $\rho_{K'}$ onto ρ_K.

Proof. It is enough to consider the elementary subdivision when a single q-dimensional cell e is divided into two q-dimensional cells e_+ and e_- introducing an additional separating $(q-1)$-dimensional cell e_0; compare [6], proof of Theorem 4.6.

We have the exact sequence of free left $\mathbb{Z}[\pi]$-chain complexes

$$0 \to C_*(\widetilde{K}) \xrightarrow{\;\psi\;} C_*(\widetilde{K'}) \to D_* \to 0 \tag{10.8}$$

where the chain complex D_* has nontrivial chains only in dimension q and $q - 1$ and D_q and D_{q-1} are both free of rank one. The free generator of the module D_q can be labelled with e_+ and the generator of D_{q-1} can be labelled with the cell e_0 and then the boundary homomorphism is given by $\partial(e_+) = e_0$. The exact sequence (10.8) induces the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}[\pi]}(D_*; M) \longrightarrow C^*(K'; M) \xrightarrow{\;\psi^*\;} C^*(K; M) \to 0. \tag{10.9}$$

Fix an admissible scalar product $\langle\,,\,\rangle$ on M. Using the cell structures of K and K', we then obtain canonically the scalar products on the complexes $\mathrm{Hom}_{\mathbb{Z}[\pi]}(D_*, M)$, $C^*(K; M)$, $C^*(K'; M)$. The chain complex $\mathrm{Hom}_{\mathbb{Z}[\pi]}(D_*, M^*)$ is acyclic and so the canonical isomorphism (7.6) identifies the determinant line of D_* with $\mathbb{R}$. The torsion of this complex equals 1. Theorem 10.7 now follows from Corollary 8.6. □

11. EXTENDED COHOMOLOGY OF AN ELLIPTIC COMPLEX

Throughout this section $\mathcal{A}$ denotes a finite von Neumann algebra with a fixed finite, normal, and faithful trace τ.

11.1. **Hilbertian $\mathcal{A}$-Bundles.** Let X be a closed manifold. A smooth bundle $p : \mathcal{E} \to X$ of topological vector spaces (cf. Chapter 3 of [19]) is called a bundle of finitely generated Hilbertian $\mathcal{A}$-modules or, simply, a Hilbertian $\mathcal{A}$-bundle if

   (i) $\mathcal{E}$ is equipped with a smooth fiberwise action $\rho : \mathcal{E} \times \mathcal{A} \to \mathcal{E}$ so that with this action each fiber $p^{-1}(x)$ is a finitely generated Hilbertian $\mathcal{A}$-module, cf. Subsection 2.5.2;

(ii) there exist a finitely generated Hilbertian $\mathcal{A}$-module M such that $p : \mathcal{E} \to X$ is locally isomorphic to the trivial bundle $p_0 : X \times M \to X$ so that the local isomorphism intertwines p, p_0 and the $\mathcal{A}$-action.

Given a Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$ over X we denote by $C^\infty(X, \mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

11.2. **Example: Flat Hilbertian $\mathcal{A}$-Bundles.** Let M be a finitely generated Hilbertian $(\pi - \mathcal{A})$-bimodule, cf. Subsection 10.1. Let X be a connected, closed, smooth manifold with fundamental group π and let $\widetilde{X}$ denote the universal covering of X. A flat Hilbertian $\mathcal{A}$-bundle with fiber M over X is an associated bundle $p : \mathcal{E} \to X$, where $\mathcal{E} = (M \times \widetilde{X})/ \sim$ with its natural projection onto X. Here $(m, x) \sim (gm, gx)$ for all $g \in \pi$, $x \in \widetilde{X}$ and $m \in M$. Then $p : \mathcal{E} \to X$ is a a bundle of finitely generated Hilbertian $\mathcal{A}$-modules.

11.3. **Elliptic complex of Hilbertian $\mathcal{A}$-modules.** Let $\mathcal{E}_0, \dots, \mathcal{E}_n$ be Hilbertian $\mathcal{A}$-modules over a closed manifold X. Consider a complex

$$0 \to C^\infty(X, \mathcal{E}_0) \xrightarrow{d_1} C^\infty(X, \mathcal{E}_1) \xrightarrow{d_1} \cdots \xrightarrow{d_n} C^\infty(X, \mathcal{E}_n) \to 0, \qquad (11.1)$$

where

$$d_j : C^\infty(X, \mathcal{E}_{j-1}) \to C^\infty(X, \mathcal{E}_j), \qquad j = 1, \dots, n, \qquad (11.2)$$

are first order differential operators such that $d_{j+1} \circ d_j = 0$.

The complex (11.1) induces in a standard way the *symbol complex* over the cotangent bundle T^*X of X. The complex (11.1) is called *elliptic* if the symbol complex is exact outside of the zero section.

The complex (11.1) is not a complex of Hilbertian spaces. Therefore, we can not directly define the the extended cohomology of this complex. However, following [13], we introduce in Subsection 11.8 the Sobolev completion of (11.1). We show that if the complex (11.1) is elliptic then the obtained extended cohomology does not depend on the choice of the Sobolev parameter s.

11.4. **Example: de Rham complex.** Given a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E} \to X$ over a closed connected manifold X one may consider the space of smooth differential j-forms on X with values in $\mathcal{E}$; this space will be denoted by $\Omega^j(X, \mathcal{E})$. It is naturally defined as a right $\mathcal{A}$-module. An element of $\Omega^j(X, \mathcal{E})$ can be uniquely represented as a π-invariant differential form on $\tilde{X}$ with values in M, i.e. as a π-invariant element of $M \otimes_{\mathbb{C}} \Omega^j(\widetilde{X})$. Here one considers the total (diagonal) π action, that is, the tensor product of the actions of π on $\Omega^j(\widetilde{X})$ and on M. More precisely, if $\omega \in \Omega^j(\widetilde{X})$ and $m \in M$, then $m \otimes \omega$ is said to be π-invariant if $gm \otimes g^*\omega = m \otimes \omega$ for all $g \in \pi$.

A flat $\mathcal{A}$-linear connection on a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$ is defined as an $\mathcal{A}$-homomorphism

$$\nabla = \nabla_{j+1} : \Omega^j(X, \mathcal{E}) \longrightarrow \Omega^{j+1}(X, \mathcal{E})$$

such that

$$\nabla(f\omega) = df \wedge \omega + f\nabla(\omega) \qquad \text{and} \qquad \nabla^2 = 0$$

for any $\mathcal{A}$ valued smooth function f on X and for any $\omega \in \Omega^j(X, \mathcal{E})$. On a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$, as defined in Subsection 11.2, there is a canonical flat $\mathcal{A}$-linear connection ∇ which is given as follows: under the identification of $\Omega^j(X, \mathcal{E})$ given in the previous paragraph, one defines the connection ∇ to be the de Rham exterior derivative. If ∇ is a flat connection on $\mathcal{E}$ then the de Rham complex

$$0 \longrightarrow \Omega^0(X, \mathcal{E}) \xrightarrow{\nabla_1} \Omega^1(X, \mathcal{E}) \xrightarrow{\nabla_2} \cdots \xrightarrow{\nabla_n} \Omega^n(X, \mathcal{E}) \longrightarrow 0$$

is an elliptic complex.

11.5. **The Laplacian.** A Hermitian metric h on a flat Hilbertian $\mathcal{A}$-bundle $p : \mathcal{E} \to X$ is a smooth family of admissible scalar products on the fibers. Let us fix a smooth measure μ on X. Then any Hermitian metric on $p : \mathcal{E} \to X$ defines the L^2 scalar product

$$\langle s_1, s_2 \rangle := \int_X h(s_1(x), s_2(x)) \, d\mu(x), \qquad s_1, s_2 \in C^\infty(X, \mathcal{E}), \tag{11.3}$$

on $C^\infty(X, \mathcal{E})$. We denote by $L^2(X, \mathcal{E})$ the completion of the space $C^\infty(X, \mathcal{E})$ with respect to the scalar product (11.3).

Suppose we are given an elliptic complex (11.1) of Hilbertian $\mathcal{A}$-modules. Assume further that each bundle $\mathcal{E}_j$, $j = 0, \ldots, n$ is endowed with a Hermitian metric h_j.

The *Laplacian* Δ_j is defined to be

$$\Delta_j = d_{j-1} d_{j-1}^* + d_j^* d_j : L^2(X, \mathcal{E}_j) \longrightarrow L^2(X, \mathcal{E}_j),$$

where d_j^* denotes the formal adjoint of d_j with respect to the L^2 scalar product on $L^2(X, \mathcal{E}_j)$. Note that, by definition, the Laplacian is a formally self-adjoint operator which is densely defined. Since it is elliptic it is also essentially self-adjoint. We denote by Δ_j the self adjoint extension of the Laplacian.

11.6. **Spectral cut-off of an elliptic complex.** For every $\mathcal{I} \subset \mathbb{R}$ we denote by

$$L_\mathcal{I}^2(X, \mathcal{E}_j) \subset L^2(X, \mathcal{E}_j)$$

the image of the spectral projection $P_\mathcal{I}^j : L^2(X, \mathcal{E}_j) \to L^2(X, \mathcal{E}_j)$ of Δ_j corresponding to $\mathcal{I}$. The following theorem was proven by M. Shubin [32, Th. 5.1][6].

Theorem 11.7. *Fix $\varepsilon > 0$. Then*
   (i) $L_{[0,\varepsilon]}^2(X, \mathcal{E}_j) \subset C^\infty(X, \mathcal{E}_j)$, *i.e.,* $L_{[0,\varepsilon]}^2(X, \mathcal{E}_j)$ *consists of smooth forms.*
   (ii) *The Hilbertian $\mathcal{A}$-module $L_{[0,\varepsilon]}^2(X, \mathcal{E}_j)$ is finitely generated, i.e., belongs to the category $\mathcal{H}_\mathcal{A}^f$, cf. Subsection 2.5.2.*

---

[6]In [32] the theorem is stated for the de Rham complex of a flat Hilbertian $\mathcal{A}$-bundle. But the proof given there is valid for a general elliptic complex.

11.8. **The Sobolev complex.** Let $W^s(X, \mathcal{E}_j)$ denote the completion of the space of smooth sections in the topology defined by the inner product

$$\langle \omega_1, \omega_2 \rangle_s = \big\langle (I + \Delta_j)^s \omega_1, \omega_2 \big\rangle, \qquad \omega_1, \omega_2 \in C^\infty(X, \mathcal{E}_j). \tag{11.4}$$

Fix $s > n$ and consider the following Sobolev complex

$$0 \rightarrow W^s(X, \mathcal{E}_0) \xrightarrow{\ d_1\ } W^{s-1}(X, \mathcal{E}_1) \xrightarrow{\ d_2\ } \cdots$$
$$\cdots \xrightarrow{\ d_n\ } W^{s-n}(X, \mathcal{E}_n) \rightarrow 0. \tag{11.5}$$

This is a complex of Hilbertian $\mathcal{A}$-modules. Denote by $\mathcal{H}^*_s(\mathcal{E}_*)$ the extended cohomology of this complex.

Let

$$\Delta_{j,s} : W^{s-j}(X, \mathcal{E}_j) \rightarrow W^{s-j}(X, \mathcal{E}_j), \qquad j = 0, \ldots, n,$$

denote the Laplacian of the complex (11.5), defined as in (11.4) but using the scalar products (11.4). Clearly,

$$\Delta_{j,s} = (I + \Delta_j)^{-1} \Delta_j. \tag{11.6}$$

Let $W^{s-j}_{[0,\delta]}(X, \mathcal{E}_j)$ denote the image of the spectral projection of $\Delta_{j,s}$ corresponding to the interval $[0, \delta] \subset \mathbb{R}$. From (11.6) we conclude

$$L^2_{[0,\varepsilon]}(X, \mathcal{E}_j) = W^{s-j}_{[0,\delta]}(X, \mathcal{E}_j), \qquad \text{where} \quad \varepsilon = \delta(1 + \delta). \tag{11.7}$$

11.9. **Extended cohomology of an elliptic complex.** Fix $\delta > 0$ and set $\varepsilon = \delta(1 + \delta)$. By Theorem 11.7, the subcomplex

$$0 \rightarrow L^2_{[0,\varepsilon]}(X, \mathcal{E}_0) \xrightarrow{\ d_1\ } L^2_{[0,\varepsilon]}(X, \mathcal{E}_1) \xrightarrow{\ d_2\ } \cdots \xrightarrow{\ d_n\ } L^2_{[0,\varepsilon]}(X, \mathcal{E}_n) \rightarrow 0 \tag{11.8}$$

of (11.1) is a complex of finitely generated Hilbertian modules.

By (11.7), the inclusion

$$L^2_{[0,\varepsilon]}(X, \mathcal{E}_j) = W^{s-j}_{[0,\delta)]}(X, \mathcal{E}_j) \hookrightarrow W^{s-j}(X, \mathcal{E}_j) \tag{11.9}$$

induces an isomorphism of the extended cohomology of the complexes (11.8) and (11.5). Thus: *the cohomology of complexes (11.5) and (11.8) depends neither on the choice of $\varepsilon > 0$ nor on the choice of the Sobolev parameter s.* We denote this cohomology by $\mathcal{H}^*(\mathcal{E}_*)$ and refer to it as the *extended cohomology of the elliptic complex* (11.1). Since $\mathcal{H}^*(\mathcal{E}_*)$ is isomorphic to the extended cohomology of the finitely generated complex (11.8), it is an object of the category $\mathcal{E}(\mathcal{H}^f_\mathcal{A})$.

11.10. **Extended de Rham cohomology.** Suppose that the elliptic complex (11.1) is the de Rham complex of a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$ with fiber M. Fix a Hermitian metric h on $\mathcal{E}$ and a Riemannian metric g on X. These data define an L^2-scalar product on $\Omega^j(X, \mathcal{E})$ in a standard way.

We denote the extended cohomology of this complex by $\mathcal{H}^*(X, \mathcal{E})$ and refer to it as the *extended de Rham cohomology of X with coefficients in $\mathcal{E}$.*

Let $W^s(\Omega^j(X, \mathcal{E}))$, $(j = 0, \ldots, n)$ denote the completion of the space $\Omega^j(X, \mathcal{E})$ of smooth forms with respect to the Sobolev scalar product (11.4). Then the complex (11.5) has the form

$$
0 \to W^s(\Omega^0(X, \mathcal{E})) \xrightarrow{\nabla} W^{s-1}(\Omega^1(X, \mathcal{E})) \xrightarrow{\nabla} \cdots
$$
$$
\cdots \xrightarrow{\nabla} W^{s-n}(\Omega^n(X, \mathcal{E})) \to 0.
\tag{11.10}
$$

Let K be a triangulation of X. Consider the cochain complex $C^*(K; M)$, cf. (10.1). If $s > 3n/2 + 1$, then the complex (11.10) consists of continuous forms and the de Rham integration map

$$
\theta : W^{s-*}(\Omega^*(X, \mathcal{E})) \longrightarrow C^*(K; M)
$$

is defined. By the de Rham theorem for extended L^2-cohomology, cf. [13, §7], [32], this map is a homotopy equivalence. In particular, it induces an isomorphism of the extended cohomology

$$
\theta_* : \mathcal{H}^*(X; \mathcal{E}) \xrightarrow{\sim} \mathcal{H}^*(K; M).
\tag{11.11}
$$

12. Analytic L^2-torsion without the determinant class condition

We continue to use the notation introduced in the previous section. In particular, $\mathcal{A}$ denotes a finite von Neumann algebra with a fixed finite, normal, and faithful trace τ.

12.1. Torsion of the complex $L^2_{[0,\varepsilon]}(X, \mathcal{E}_*)$.

The L^2-scalar product (11.3) on $C^\infty(X, \mathcal{E}_*)$ induces an admissible scalar product on the complex $(L^2_{[0,\varepsilon]}(X, \mathcal{E}_*), d_*)$ of finitely generated Hilbertian $\mathcal{A}$-modules. We denote by

$$
\rho_{[0,\varepsilon]} := \phi(\sigma) \in \det \mathcal{H}^*(\mathcal{E}_*).
\tag{12.1}
$$

its torsion as defined in Definition 7.5.

12.2. Torsion of the complex $L^2_{(\varepsilon,\infty)}(X, \mathcal{E}_*)$.

The complex $(L^2_{(\varepsilon,\infty)}(X, \mathcal{E}_*), d_*)$ is acyclic and one would like to define its torsion using the formula (7.10). However, since there is no trace on the category $\mathcal{H}_\mathcal{A}$ we need to use the ζ-function regularization of the determinant.

Let us recall the basic facts about the ζ-function of an elliptic operator on a Hilbertian $\mathcal{A}$-bundle, cf. [5]. We will use the well known fact that for each $\varepsilon \geq 0$ the heat operators

$$
e^{-t\Delta_j|_{L^2_{(\varepsilon,\infty)}(X, \mathcal{E}_j)}}, \qquad j = 0, \ldots, n,
$$

have smooth Schwartz kernels which are smooth sections of a bundle over $X \times X$ with fiber the commutant of M, cf. [5, 18, 24]. The symbol Tr_τ denotes application of the canonical trace on the commutant (cf. Subsection 2.8) to the restriction of the kernels to the diagonal followed by integration over the manifold X. The trace Tr_τ vanishes on commutators of smoothing operators. See also [25, 20, 18] for the case of the de Rham complex of a flat bundle defined by the regular representation of the fundamental group.

It is shown in [5, §2.4] that the integral

$$
\zeta^j_{(\varepsilon,\infty)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \, \mathrm{Tr}_\tau\big(e^{-t\Delta_j|_{L^2_{(\varepsilon,\infty)}(X, \mathcal{E}_j)}}\big) \, dt,
\tag{12.2}
$$

converges for $\Re s < -\dim X/2$. Moreover, the function $\zeta^j_{(\varepsilon,\infty)}(s)$ is analytic in s for $\Re s < -\dim X/2$ and has a meromorphic extension to the whole complex plane which is regular at $s = 0$.

Set

$$\log \operatorname{Det} \Delta_j|_{L^2_{(\varepsilon,\infty)}(X,\mathcal{E}_j)} \;=\; \frac{d}{ds}|_{s=0}\, \zeta^j_{(\varepsilon,\infty)}(s). \tag{12.3}$$

Lemma 12.3. *For every $\varepsilon_1 > \varepsilon_2 > 0$ we have*

$$\operatorname{Det} \Delta_j|_{L^2_{(\varepsilon_2,\infty)}(X,\mathcal{E}_j)} \;=\; \operatorname{Det} \Delta_j|_{L^2_{(\varepsilon_1,\infty)}(X,\mathcal{E}_j)} \cdot \operatorname{Det}_\tau \Delta_j|_{L^2_{(\varepsilon_2,\varepsilon_1]}(X,\mathcal{E}_j)}, \tag{12.4}$$

where $\operatorname{Det}_\tau \Delta_j|_{L^2_{(\varepsilon_2,\varepsilon_1]}(X,\mathcal{E}_j)}$ is the Fuglede-Kadison determinant of the automorphism Δ_j of the finitely generated $\mathcal{A}$-module $L^2_{(\varepsilon_2,\varepsilon_1]}(X,\mathcal{E}_j)$.

Proof. This easily follows from the definitions using formula (7.10). □

We now define the torsion $\rho_{(\varepsilon,\infty)}$ using (7.10) and (12.3), i.e., we set

$$\log \rho_{(\varepsilon,\infty)} \;:=\; \frac{1}{2} \sum_{j=0}^{n} (-1)^j j \frac{d}{ds}|_{s=0}\, \zeta^j_{(\varepsilon,\infty)}(s). \tag{12.5}$$

From Lemma 12.3 we obtain:

Lemma 12.4. *The product*

$$\rho_{[0,\varepsilon]} \cdot \rho_{(\varepsilon,\infty)} \;\in\; \det \mathcal{H}^*(\mathcal{E}_*)$$

is independent of $\varepsilon > 0$.

Lemma 12.4 allows us to give the following definition.

Definition 12.5. *The analytic L^2 torsion of the complex $(C^\infty(X,\mathcal{E}_*), d_*)$ is defined by the formula*

$$\rho_\mathcal{E} \;:=\; \rho_{[0,\varepsilon]} \cdot \rho_{(\varepsilon,\infty)} \;\in\; \det \mathcal{H}^*(\mathcal{E}_*). \tag{12.6}$$

The L^2-torsion of the de Rham complex of a flat Hilbertian $\mathcal{A}$-bundle $\mathcal{E}$, is called the analytic L^2-torsion of bundle $\mathcal{E}$.

Note that, in general, the torsion $\rho_\mathcal{E}$ depends on the choices of the hermitian metrics on $\mathcal{E}_i$ and the measure on X. We shall show below in Section 13 that the de Rham torsion is independent of these choices if $\dim X$ is odd and the bundle $\mathcal{E}$ is unimodular.

13. The Cheeger-Müller type theorem

In this section we recall the definition of the relative torsion of a flat unimodular Hilbertian $\mathcal{A}$-bundle over a compact Riemannian manifold, see [3], [8]. We show that the relative torsion equals to an appropriately defined ratio of the de Rham and combinatorial torsions. Burghelea, Friedlander and Kappeler [3] proved that the relative torsion equals one for odd dimensional manifolds. Using this result we prove here that for odd dimensional manifolds the analytic L^2 torsion coincides with the combinatorial L^2 torsion, where both these

torsions are viewed as elements of the determinant line of the extended L^2 cohomology. Previously this result was proven in [5] under an additional assumption that the Hilbertian bundle is of determinant class.

13.1. **The setting.** Let X be a closed smooth manifold of dimension $\dim X = n$ with fundamental group $\pi = \pi_1(X)$. Let $\mathcal{A}$ be a finite von Neumann algebra and let M be a finitely generated unimodular Hilbertian $(\pi - \mathcal{A})$-bimodule, cf. Subsection 10.1. Let $\mathcal{E}$ be the flat Hilbertian $\mathcal{A}$-bundle with fiber M over X, cf. Subsection 11.2. Fix a Hermitian metric h on $\mathcal{E}$ (cf. Subsection 11.5) and a Riemannian metric g on X. Let $\rho_{\mathcal{E}} \in \det \mathcal{H}^*(X; \mathcal{E})$ be the analytic L^2-torsion, cf. Definition 12.5.

Fix a cell decomposition K of X and a positive element of $\det M$. Let $\rho_{K,M} \in \det \mathcal{H}^*(K; M)$ denote the combinatorial L^2-torsion, cf. Definition 10.5.

13.2. **The relative torsion.** Fix a Sobolev index $s > n/2 + 1$. Then the Sobolev space $W^s(\Omega^*(X, \mathcal{E}))$ (cf. Subsection 11.10) consists of continuous forms and the de Rham integration map

$$\theta : W^s(\Omega^*(X, \mathcal{E})) \longrightarrow C^*(K; M)$$

is defined. Consider the following sequence of chain maps:

$$(\Omega^*(X, \mathcal{E}), \nabla) \xrightarrow{(\Delta + I)^{-s/2}} (W^s(\Omega^*(X, \mathcal{E}), \nabla) \xrightarrow{\theta_s} (C^*(K; M), \partial).$$

The composition of these maps is denoted by $g(s, K)$. We form the cone complex:

$$\mathrm{Cone}(g(s, K)) = C^*(K, M) \oplus \Omega^{*-1}(X, \mathcal{E})$$

with differential

$$D_i = D_i(s, K) = \begin{pmatrix} -\nabla_i & 0 \\ g(s, K)_{i-1} & \partial_{i-1} \end{pmatrix}, \tag{13.1}$$

where ∂ is the coboundary map of the complex $C^*(K; M)$. It follows (as in [8, §4], see also [3, §2]) that the cone complex is acyclic. Hence the torsion of the cone complex is simply a positive real number, which is defined as follows.

Set

$$\Delta_j := D_{j-1}D_{j-1}^* + D_j^* D_j.$$

As in Subsection 12.2, we can define the trace Tr_τ of the heat kernel $e^{-t\Delta_j}$ and the ζ-function

$$\zeta_{\Delta_j}(\sigma) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} \, \mathrm{Tr}_\tau(e^{-t\Delta_j}) \, dt.$$

It is shown in [8, §6] (cf. also [3, §2]) that this function is defined and analytic for $\Re \sigma > n/2$ and has a meromorphic continuation to the whole complex plane which is regular at $\sigma = 0$.

Definition 13.3. *The relative torsion* $\mathcal{R} = \rho_{\mathrm{Cone}(g(s,K))}$ *is defined by the formula*

$$\log \mathcal{R} = \frac{1}{2} \sum_j (-1)^j j \zeta'_{\Delta_j}(0).$$

13.4. **Relations between the de Rham, the combinatorial, and the relative torsions.** The de Rham integration map (11.11) induces an isomorphism

$$\theta_* : \det \mathcal{H}^*(X, E) \ \overset{\sim}{\longrightarrow} \ \det \mathcal{H}^*(K; M). \tag{13.2}$$

We denote by

$$\overline{\theta} : \det \mathcal{H}^*(X, E) \otimes (\det \mathcal{H}^*(K; M))^* \ \overset{\sim}{\longrightarrow} \ \mathbb{R}, \tag{13.3}$$

the isomorphism induced by θ_*.

Let

$$\rho_{K,M}^* \ \in \ (\det \mathcal{H}^*(K; M))^*$$

be the dual of the combinatorial torsion $\rho_{K,M}$. It is the unique element of $(\det \mathcal{H}^*(K; M))^*$ such that $\langle \rho_{K,M}^*, \rho_{K,M} \rangle = 1$; here $\langle , \rangle$ denotes the pairing between $(\det \mathcal{H}^*(K; M))^*$ and $\det \mathcal{H}^*(K; M)$.

Theorem 13.5. *One has*

$$\mathcal{R} \ = \ \overline{\theta}\big(\rho_{\mathcal{E}} \otimes \rho_{K,M}^* \big). \tag{13.4}$$

Proof. Let $L^2 \Omega^j(X, \mathcal{E})$ denote the space of square integrable differential forms on X with values in $\mathcal{E}$. For every $\mathcal{I} \subset \mathbb{R}$ we denote by $\Omega_{\mathcal{I}}^j(X, \mathcal{E})$ the intersection of the image of the spectral projection $P_{\mathcal{I}}^j : L^2 \Omega^i(X, \mathcal{E}) \to L^2 \Omega^j(X, \mathcal{E})$ of the Laplacian $\Delta_j = \nabla_{i-1} \nabla_{i-1}^* + \nabla_i^* \nabla_i$ corresponding to $\mathcal{I}$ with the space of smooth forms. By Theorem 11.7 the image of the spectral projection corresponding to $[0, \varepsilon]$ consists of smooth forms. It follows that $\Omega^j(X, \mathcal{E}) = \Omega_{[0,\varepsilon]}^j(X, \mathcal{E}) \oplus \Omega_{(\varepsilon,\infty)}^j(X, \mathcal{E})$ for every $\varepsilon > 0$.

Set

$$g_{\mathcal{I}}(s, K)_j \ := \ g(s, K)_j |_{\Omega_{\mathcal{I}}^j(X, \mathcal{E})}.$$

Then, for every $\varepsilon > 0$, we obtain an exact sequence of acyclic complexes

$$0 \ \to \ \text{Cone}(g_{[0,\varepsilon]}(s, K)) \ \overset{j}{\longrightarrow} \ \text{Cone}(g(s, K)) \ \overset{\pi}{\longrightarrow} \ \Omega_{(\varepsilon,\infty)}^*(X, \mathcal{E}) \ \to \ 0,$$

where j is the natural inclusion and π is the natural projection. The Carey-Mathai-Mishchenko lemma, cf. [8], [3, Lemma 1.14], implies that

$$\mathcal{R} = \rho_{\text{Cone}(g(s,K))} \ = \ \rho_{\Omega_{(\varepsilon,\infty)}^*}(X,\mathcal{E}) \cdot \rho_{\text{Cone}(g_{[0,\varepsilon]}(s,K))}. \tag{13.5}$$

From Corollary 8.6 we conclude that

$$\rho_{\text{Cone}(g_{[0,\varepsilon]}(s,K))} \ = \ \overline{\theta}(\rho_{\Omega_{[0,\varepsilon]}^*(X,\mathcal{E})} \otimes \rho_{K,M}^*). \tag{13.6}$$

Combining (12.6), (13.5) and (13.6) we obtain (13.2). $\qquad \square$

Remark 13.6. In the case when the extended cohomology $\mathcal{H}(X, \mathcal{E}) \simeq \mathcal{H}(K, M)$ is trivial and, hence, the torsions $\rho_{\mathcal{E}}$ and $\rho_{K,M}$ are just real numbers, the right hand side of (13.4) is the ration of these numbers and we obtain

$$\mathcal{R} \ = \ \rho_{\mathcal{E}} / \rho_{K,M}.$$

From Theorem 13.5 it follows that the relative torsion is independent of the Sobolev parameter s (this was originally proven in [8]). Also, from (13.4) and from the invariance of

the combinatorial torsion under subdivisions we conclude that the relative torsion is independent of the cell decomposition K of X; this result is called the *combinatorial invariance* of the relative torsion and was first proven in [3].

13.7. **The odd dimensional case and the Cheeger-Müller type theorem.** Let us consider the case when the dimension n of X is odd. The Euler characteristic of K vanishes $\chi(K) = 0$ and therefore the combinatorial torsion $\rho_{K,M}$ is independent of the choice of the volume form on M, cf. Remark 10.6.1. Assume that the Hermitian metric h on $\mathcal{E}$ is unimodular. The main result of [3] (cf. Th. 0.1 in [3]) states that, in this situation the relative torsion equals one, i.e. $\mathcal{R} = 1$. Combined with Theorem 13.5 this result implies the following extension of the Cheeger-Müller theorem

Theorem 13.8. *Suppose that the dimension of X is odd and that the Hermitian metric h on $\mathcal{E}$ is unimodular. Then the isomorphism* (13.2) *identifies the analytic and the combinatorial torsions*

$$\theta_*(\rho_{\mathcal{E}}) = \rho_{K,M}. \tag{13.7}$$

In particular, the analytic L^2 torsion $\rho_{\mathcal{E}}$ does not depend on the choice of the Riemannian metric on X and the unimodular Hermitian metric on $\mathcal{E}$.

Theorem 0.1 of [3] calculates the relative torsion in the general case without assuming that the dimension of X is odd and that the Hermitian metric on $\mathcal{E}$ is unimodular. In this case $\log \mathcal{R}$ is represented as an integral over X of some explicitly calculated locally defined differential form (this form was originally found by Bismut and Zhang, [2], who considered the case of finite dimensional bundle $\mathcal{E}$). We refer the reader to [3] for details.

The fact that the relative torsion (and, hence, the analytic torsion) is independent of the metrics is much easier to prove than the equality $\mathcal{R} = 1$. Moreover, this fact is used in the proof of this equality, cf. [3]. The independence of the metrics is proven in section 4 of [3]. Again, the result of [3] is more general and calculates explicitly the dependence of $\mathcal{R}$ on the metrics in the case when the dimension of X is even (so called, *anomaly formula*). Using this result and Theorem 13.5 one can easily obtain a similar formula for the dependence of the analytic torsion on the metrics. We leave the details for an interested reader.

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